## SF2729 Groups and Rings <br> Exam <br> Monday, March 21, 2016

Time: 08:00-13:00
Allowed aids: none
Examiner: Roy Skjelnes
Present your solutions to the problems in a way such that arguments and calculations are easy to follow. Provide detailed arguments to your answers. An answer without explanation will be given no points.

The final exam consists of six problems, each of which can give up to 6 points. The homework problems will contribute with up to 9 points on the three first problems in the exam. Credits on the three first problems in the exam will together with the contribution from the homeworks give at most 18 points.

The minimum scores required for each grade are given by the following table:

| Grade | A | B | C | D | E | Fx |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: |
| Credit | 30 | 27 | 24 | 21 | 18 | 16 |

A score of 16 or 17 is a failing grade with the possibility to improve to an E grade by additional work.

## Problem 1

Let $\Phi: G \longrightarrow H$ be a group homomorphism.
(a) Show that the kernel, $\operatorname{ker} \Phi$, is a normal subgroup.
(b) Show that the image of $\Phi$ is subgroup. Is it normal?
(c) For any $a \in G$, show that the order of $\Phi(a)$ divides the order of $a$.

## Problem 2

Let $R$ be a commutative ring with 1 .
(a) Define what an nilpotent element is, and show that the set of nilpotent elements in $R$ form an ideal.
(b) Show that if an element $x \in R$ is nilpotent, then $x$ is contained in any prime ideal of $R$.
(c) Give an example of a ring $R$ that is not an integral domain, and has no nilpotent elements other than zero.

## Problem 3

(a) Describe the irreducible factors of the monic polynomial

$$
f(x)=x^{5}-x^{4}+3 x^{3}+3 x^{2}-3 x-3 \text { in } \mathbb{Q}[x] .
$$

(b) Let $F(x)=\left(x-a_{1}\right) \cdots\left(x-a_{n}\right)$ be a polynomial with distinct roots $a_{i} \neq a_{j}$ (when $i \neq j$ ) in $\mathbb{Q}$. Show that

$$
\begin{equation*}
\mathbb{Q}[x] /(F(x)) \simeq \prod_{i=1}^{n} \mathbb{Q} . \tag{3~p}
\end{equation*}
$$

## Problem 4

Let $G=\mathrm{GL}_{2}\left(\mathbb{Z}_{3}\right)$ the group of invertible $(2 \times 2)$-matrices with entries in the field with three elements. The group acts naturally on the vector space $\mathbb{Z}_{3}^{2}$, and also on the set $L$ of lines that passes through origin.
(a) Determine the order of $G$.
(b) Show that the center of $G$ is the set of scalar matrices, and isomorphic to $S_{2}$.
(c) Show that the action of $G$ on the set $L$ induces a surjective group homomorphism from $G$ to $S_{4}$; the symmetric group in four letters.

## Problem 5

Let $R=\mathbb{Z}[\sqrt{7}]=\mathbb{Z}[x] /\left(x^{2}-7\right)$, which is a free $\mathbb{Z}$-module of rank 2 . Let $N: R \rightarrow \mathbb{Z}$ be the map $N(a+b x)=a^{2}-7 b^{2}$. Then $N$ is multiplicative.
(a) Determine a unit $\xi$, different from $\pm 1$.
(b) Show that $(2+x)$ and $(2-x)$ are co-prime. ${ }^{1}$
(c) Write 6 as a product of irreducibles, and verify if any of these irreducible components are associates

## Problem 6

Let $\mathbb{Q}$ denote the rational numbers, which is an abelian group under addition.
(a) Show that $\mathbb{Q}$ is not a finitely generated group.
(b) Show that $\operatorname{Aut}(\mathbb{Q})$ is isomorphic to $\mathbb{Q}^{*}$; the multiplicative group of the non-zero rational numbers.
(c) Show that $\operatorname{Aut}\left(\mathbb{Q}^{2}\right)$ is isomorphic to $\mathrm{GL}_{2}(\mathbb{Q})$.

[^0]
[^0]:    ${ }^{1}$ Co-prime is the same as co-maximal.

