SF1625 Calculus in one variable
Exam
Tuesday March 22nd 2016

Time: 08:00-13:00
No calculators, formula sheets etcetera allowed
Examiner: Lars Filipsson

This exam consists of nine problems, each worth four points, hence the maximal score is 36. Part A consists of the three first problems. To the score on part A your bonus points are added, up to a maximum of 12. The score on part A is at most 12, bonus points included. The bonus points are added automatically. You can check how many bonus points you have on your results page.

The following three problems constitute part B and the last three problems part C. You need a certain amount of points from part C to obtain the highest grades.

The grading will be performed according to this table:

Grade	A	B	C	D	E	Fx
Total score	27	24	21	18	16	15
score on part C	6	3	–	–	–	–

To obtain a maximal 4 for a solution to a problem on the exam, your solution must be well presented and easy to follow. Notation must be explained, the logical structure of the solution must be clearly described in words or in symbols and the reasoning leading up to the conclusion must be well motivated and clearly explained. Solutions that are clearly inadequate in these respects will be awarded no more than 2 points.

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PART A

1. The 1st of January 2006 10 kg of a certain radioactive substance was locked up in a basement. The substance decays at a rate proportional to the amount of the substance present, with a half life of 50 years. How much of the substance remains on January 1st 2016?

2. Compute the integrals:

   A. $\displaystyle\int_0^{2\pi} |\sin x + \cos x|\,dx$ (hint: split the interval of integration)

   B. $\displaystyle\int_1^{e} x^2 \ln x\,dx$ (hint: use integration by parts)

3. We study the function f given by $f(x) = \dfrac{x+2}{x^2+1} + 2\arctan x$.

   A. Determine the domain of definition of f.
   B. Find the intervals where f is increasing and decreasing, respectively.
   C. Determine whether f assumes maximum and a minimum values.
   D. Find all asymptotes to the graph $y = f(x)$
   E. Using the above, find the range of f.

PART B

4. We shall study Taylor approximation of the function $f(x) = \ln(1+x)$.
   A. Find the Taylor polynomial of degree 4 around the point $x = 0$ to f.
   B. Use the polynomial from A to obtain an approximate value of $\ln 2$.
   C. Decide whether the error of your approximation is less than 0.25.

5. We shall find the center of mass (x_T, y_T) of the upper half of the unit disc, i.e. the area given by $x^2 + y^2 \leq 1$ and $y \geq 0$. For symmetry reasons $x_T = 0$, but the y-coordinate needs to be calculated. One can show that

$$y_T = \frac{\displaystyle\int_0^1 2y\sqrt{1-y^2}\,dy}{\displaystyle\int_0^1 2\sqrt{1-y^2}\,dy}.$$

   Compute the y-coordinate of the center of mass!

6. An object with mass m is falling towards the surface of the earth. If we assume the air resistance to be proportional to the speed v, we obtain from Newton's sedond law the differential equation

$$mv'(t) = -kv(t) + mg$$

   where k is a positive constant and g is the gravitational acceleration.
   A. Find the speed v at an arbitrary time t, if the object is released with zero speed at the time $t = 0$.
   B. Show that the speed cannot increase without bound and in fact will approach a limiting value after long time. Find this limiting value.

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Part C

7. This problem is about the theory of local extreme points.
   A. Define what is meant by a local maximum of a function f.
   B. Prove this statement: If the function f assumes a local maximum at an interior point a in the domain of definition and f is differentiable at a, then $f'(a) = 0$.
   C. Give an example showing that a function can have a derivative that is 0 at a point without assuming a local extreme value at that point.
   D. Give an example showing that a function can assume a local maximum at a point without having a derivative that is zero at that point.

8. We study the curve $y = x^4$. For each point (x, y) on the curve (except for the origin) the curve has a normal line intersecting the y-axis at exactly one point $(0, b)$. Find the smallest possible value of b.

9. Determine whether the improper integral
$$\int_0^{\pi} \frac{dx}{x \sin x + \sqrt{x}}$$
is convergent or divergent.