KTH Teknikvetenskap

SF1625 Envariabelanalys
Solutions to exam 22 march 2016

PART A

1. The 1st of January 2006 10 kg of a certain radioactive substance was locked up in a basement. The substance decays at a rate proportional to the amount of the substance present, with a half life of 50 years. How much of the substance remains on January 1st 2016?

Solution. If $y(t)$ describes the amount of the substance at time t, where $t = 0$ is 1st of january 2006, then $y'(t) = ky(t)$ for some constant k and consequently $y(t) = Ce^{kt}$ where C is another constant.

At time 0 there is 10 kg of the substance, and so $C = 10$.

We know that at time $t = 50$ we have half of the original amount, that is $y(50) = 5$. We get:

$$\begin{aligned} y(50) = 5 &\iff 10e^{50k} = 5 \\ &\iff e^{50k} = \frac{1}{2} \\ &\iff 50k = \ln\frac{1}{2} \\ &\iff k = \frac{\ln\frac{1}{2}}{50} = -\frac{\ln 2}{50}. \end{aligned}$$

The amount at time t is therefore $y(t) = 10e^{-(t\ln 2)/50}$ kg. January 1st 2016 this is precisely $y(10) = 10e^{-(10\ln 2)/50} = 10e^{-(\ln 2)/5} \approx 8.7$ kg.

□

Answer: $10e^{-(\ln 2)/5}$ kg

2. Compute the integrals:

A. $\displaystyle\int_0^{2\pi} |\sin x + \cos x|\,dx$ (hint: split the interval of integration)

B. $\displaystyle\int_1^{e} x^2 \ln x\,dx$ (hint: use integration by parts)

Solution. A. Since $\sin x + \cos x$ is positive when $0 < x < 3\pi/4$, negative when $3\pi/4 < x < 7\pi/4$ and positive when $7\pi/4 < x < 2\pi$ we obtain

$$\begin{aligned}
&\int_0^{2\pi} |\sin x + \cos x|\,dx \\
&= \int_0^{3\pi/4} (\sin x + \cos x)\,dx - \int_{3\pi/4}^{7\pi/4} (\sin x + \cos x)\,dx + \int_{7\pi/4}^{2\pi} (\sin x + \cos x)\,dx \\
&= 4\sqrt{2}.
\end{aligned}$$

B. We use integration by parts and obtain:

$$\begin{aligned}
\int_1^e x^2 \ln x\,dx &= \left[\frac{x^3}{3}\ln x\right]_1^e - \int_1^e \frac{x^2}{3}\,dx \\
= \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9} = \frac{2e^3+1}{9}
\end{aligned}$$

□

Answer: A. $4\sqrt{2}$. B. $\frac{2e^3+1}{9}$

3. We study the function f given by $f(x) = \dfrac{x+2}{x^2+1} + 2\arctan x$.

   A. Determine the domain of definition of f.
   B. Find the intervals where f is increasing and decreasing, respectively.
   C. Determine whether f assumes maximum and a minimum values.
   D. Find all asymptotes to the graph $y = f(x)$
   E. Using the above, find the range of f.

Solution. A. The domain is $\mathbf{R}$. Since $\lim_{x\to\pm\infty} f(x) = \pm\pi$ we have the asymptote $y = \pi$ at ∞ and the asymptote $y = -\pi$ at $-\infty$.

B.We differentiate and obtain

$$f'(x) = \frac{x^2+1-2x(x+2)}{(x^2+1)^2} + \frac{2}{x^2+1} = \frac{x^2-4x+3}{(x^2+1)^2}.$$

We get two critical points, namely $x = 3$ and $x = 1$. We study the sign of the derivative:
If $x < 1$ then $f'(x)$ is positive.
If $1 < x < 3$ then $f'(x)$ is negative.
If $x > 3$ the $f'(x)$ is positive.

It follows that f is increasing on the interval $x \leq 1$, decreasing on the interval $1 \leq x \leq 3$ and increasing on the interval $x \geq 3$

C, D. It follows from the above that f has a local max at $x = 1$ (and $f(1) = (3+\pi)/2$) and a localt min at $x = 3$ (and $f(3) = 1/2 + 2\arctan 3$). These local extreme values are strictly between $-\pi$ and π and hence cannot be global extreme values, because f assumes values arbitrarily close to π and $-\pi$.

To summarize, f does not assume a global max or a global min. The range is $(-\pi, \pi)$, because f is continuous and assumes values arbitrarily close to $\pm\pi$.

□

Answer: Se lösningen.

PART B

4. We shall study Taylor approximation of the function $f(x) = \ln(1+x)$.
   A. Find the Taylor polynomial of degree 4 around the point $x = 0$ to f.
   B. Use the polynomial from A to obtain an approximate value of $\ln 2$.
   C. Decide whether the error of your approximation is less than 0.25.

Solution. A. $p(x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4}$. This is standard.

B. $\ln 2 = f(1) \approx p(1) = 7/12 \approx 0.6$.

C. The error is $\dfrac{f^{(5)(c)}}{5!} 1^5$ for some c between 0 and 1, and the absolute value of this is less than $1/5$ because $f^{(5)}(c) = 4!/(1+c)^4$. Answer yes.

□

5. We shall find the center of mass (x_T, y_T) of the upper half of the unit disc, i.e. the area given by $x^2 + y^2 \leq 1$ and $y \geq 0$. For symmetry reasons $x_T = 0$, but the y-coordinate needs to be calculated. One can show that

$$y_T = \frac{\displaystyle\int_0^1 2y\sqrt{1-y^2}\,dy}{\displaystyle\int_0^1 2\sqrt{1-y^2}\,dy}.$$

Compute the y-coordinate of the center of mass!

Solution. The integral in the denominator gives precisely the area of the half disc, and so that integral must equal $\pi/2$ (this can also be computed, using the substitution $y = \sin x$).

The integral in the numerator can be computed thus:

$$\int_0^1 2y\sqrt{1-y^2}\,dy = \left[\frac{-(1-y^2)^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}.$$

We see that the y-coordinate of the center of mass is $\dfrac{4}{3\pi}$

□

Answer: $4/3\pi$

6. An object with mass m is falling towards the surface of the earth. If we assume the air resistance to be proportional to the speed v, we obtain from Newton's sedond law the differential equation

$$mv'(t) = -kv(t) + mg$$

where k is a positive constant and g is the gravitational acceleration.

A. Find the speed v at an arbitrary time t, if the object is released with zero speed at the time $t = 0$.

B. Show that the speed cannot increase without bound and in fact will approach a limiting value after long time. Find this limiting value.

Solution. We can write the differential equation

$$\frac{dv}{dt} + \frac{k}{m}v = g$$

The solution v has the structure $v = v_h + v_p$ where v_h is the general solution to the corresponding homogeneous equation (i.e. with right-hand-side 0) and v_p is any particular solution. We obviously can take $v_p = gm/k$. In order to find v_h we see that the characteristic equation $r + (k/m) = 0$ has solution $r = -k/m$ and hence $v_h = Ce^{-kt/m}$. Putting all this together we get:

$$v(t) = Ce^{-kt/m} + \frac{gm}{k}, \qquad C \text{ an arbitrary constant,}$$

is the general solution to the differential equation. If the object is released with zero speed then $v(0) = 0$ and so we must choose $C = -gm/k$.

The speed at an arbitrary time t is therefore given by

$$v(t) = \frac{-gm}{k}e^{-kt/m} + \frac{gm}{k} = \frac{gm}{k}\left(1 - e^{-kt/m}\right).$$

B. Since $e^{-kt/m}$ is decreasing, v is increasing, and since $\lim_{t\to\infty} v(t) = gm/k$ this is the limiting value.

□

Answer: Se lösningen.

PART C

7. This problem is about the theory of local extreme points.
   A. Define what is meant by a local maximum of a function f.
   B. Prove this statement: If the function f assumes a local maximum at an interior point a in the domain of definition and f is differentiable at a, then $f'(a) = 0$.
   C. Give an example showing that a function can have a derivative that is 0 at a point without assuming a local extreme value at that point.
   D. Give an example showing that a function can assume a local maximum at a point without having a derivative that is zero at that point.

Solution. A. See Definition 2 of chapter 4.4 in the book.
B. See Theorem 14 of chapter 2.8 in the book.

C. The function $f(x) = x^3$ satisfies $f'(0) = 0$ but does not assume a local extreme value at $x = 0$

D. The function $g(x) = -|x|$ assumes a local max at the origin but is not differentiable there.

□

Answer: Se lösningen

8. We study the curve $y = x^4$. For each point (x, y) on the curve (except for the origin) the curve has a normal line intersecting the y-axis at exactly one point $(0, b)$. Find the smallest possible value of b.

Solution. Put $f(x) = x^4$. Since f is even it is enough to study the problem for positive x. We differentiate and obtain $f'(x) = 4x^3$. The slope of the normal to the curve $y = x^4$ at (x_0, x_0^4) is given by $-1/4x_0^3$. The equation of the normal line is

$$y - x_0^4 = -\frac{1}{4x_0^3}(x - x_0).$$

This line cuts the y-axis at the point

$$(0, x_0^4 + \frac{1}{4x_0^2}.$$

Hence we want to minimize the function $d(x) = x^4 + \dfrac{1}{4x^2}$ då $x > 0$. We differentiate and get

$$d'(x) = 4x^3 - \frac{1}{2x^3}.$$

For positive x we get

$$d'(x) = 0 \Longleftrightarrow x^6 = \frac{1}{8} \Longleftrightarrow x = \frac{1}{\sqrt{2}}$$

Observing that $d'(x)$ is negative when $0 < x < 1/\sqrt{2}$ and positive when $x > 1/\sqrt{2}$ we see that this points gives the minimum value.

The minimum value of b is $b = d(1/\sqrt{2}) = 3/4$.

□

Answer: $3/4$

9. Determine whether the improper integral

$$\int_0^{\pi} \frac{dx}{x \sin x + \sqrt{x}}$$

is convergent or divergent.

Solution. Integral is improper at $x = 0$ because the integrand is not bounded there. We have

$$0 \leq \frac{1}{x \sin x + \sqrt{x}} \leq \frac{1}{\sqrt{x}} \text{då } 0 < x < \pi,$$

since $x \sin x \geq 0$. Since furthermore

$$\int_0^{\pi} \frac{dx}{\sqrt{x}} = \lim_{c \to 0^+} \int_c^{\pi} \frac{dx}{\sqrt{x}} = \lim_{c \to 0^+} (2\sqrt{\pi} - 2\sqrt{c}) = 2\sqrt{\pi}$$

it follows that

$$\int_0^{\pi} \frac{dx}{x \sin x + \sqrt{x}}$$

is convergent.

□

Answer: Convergent
