H

(Incompressible) Navier–Stokes equations

$$\begin{cases} \rho(\dot{u} + u\cdot\nabla u) = -\nabla p + \mu\Delta u + \rho f \\ \nabla\cdot u = 0 \end{cases}$$

Conservation of momentum (Newton 2nd Law)
Conservation of mass (divergence free flow)

$$\begin{cases} \dot{u} + u\cdot\nabla u = -\frac{1}{\rho}\nabla p + \nu\Delta u + f \\ \nabla\cdot u = 0 \end{cases}$$

kinematic viscosity $\nu = \frac{\mu}{\rho}$
dynamic viscosity μ
density ρ

- Choose characteristic length scale L
- Choose characteristic speed U
- This determines a time scale $\tau = \frac{L}{U}$

Introduce new dimensionless variables

$$\tilde{x} = \frac{x}{L} \quad \tilde{t} = \frac{t}{\tau} \quad \tilde{u} = \frac{u}{U}$$

$$\tilde{\nabla} = L\nabla \quad \tilde{\Delta} = L^2\Delta$$

$$\begin{cases} \dfrac{U}{\tau}\dot{\tilde{u}} + \dfrac{U^2}{L}(\tilde{u}\cdot\tilde{\nabla})\tilde{u} = -\dfrac{1}{\rho}\dfrac{1}{L}\tilde{\nabla}p + \dfrac{\nu U}{L^2}\tilde{\Delta}\tilde{u} \\ \dfrac{U}{L}\tilde{\nabla}\cdot\tilde{u} = 0 \end{cases}$$

- Multiply by $\dfrac{L}{U^2}$ and set $\tilde{p} = \dfrac{1}{\rho U^2}p$

$$\begin{cases} \dot{u} + u\cdot\nabla u = -\nabla p + \dfrac{\nu}{LU}\Delta u \\ \nabla\cdot u = 0 \end{cases}$$

($\sim$ dropped)

- Reynolds number $Re = \dfrac{LU}{\nu}$

$$\begin{cases} \dot{u} + u\cdot\nabla u = -\nabla p + Re^{-1}\Delta u \\ \nabla\cdot u = 0 \end{cases}$$

$Re \to \infty \Rightarrow$ The Euler equations

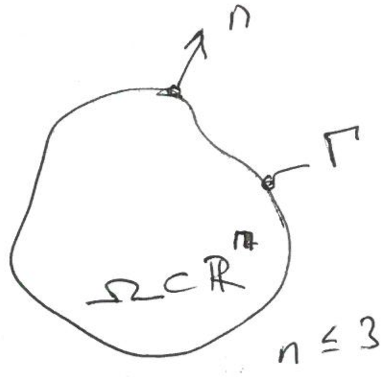
$$\begin{cases} \dot{u} + u\cdot\nabla u + \nabla p = 0 \\ \nabla\cdot u = 0 \end{cases}$$

$Re \to 0$, $p = \rho\dfrac{\nu U}{L}\tilde{p} \Rightarrow$ The Stokes eqns.

$$\begin{cases} -\Delta u + \nabla p = f \\ \nabla\cdot u = 0 \end{cases}$$

L1

Stokes problem



$$\begin{cases} -\Delta u + \nabla p = f & x \in \Omega \\ \nabla \cdot u = 0 & x \in \Omega \end{cases}$$

$$\begin{cases} u = g & x \in \Gamma_D \\ -\nabla u \cdot n + p n = h & x \in \Gamma_N \end{cases}$$

- Pressure determined only up to a constant $(\Gamma = \Gamma_D)$
  ($p + C$ also a solution)

- Add normalizing condition $\int_\Omega p \, dx = 0$

Variational formulation: introduce function spaces

$$V = [H_0^1(\Omega)]^n \quad Q = L_0^2 = \{ q \in L^2(\Omega) : \int_\Omega q \, dx = 0 \}$$

$$L^2(\Omega) = \{ v : \int_\Omega |v|^2 dx < \infty \}$$

$$H^1(\Omega) = \{ v : \int_\Omega |v|^2 + |\nabla v|^2 dx < \infty \}$$

$$H_0^1(\Omega) = \{ v : v \in H^1(\Omega), v = 0 \text{ on } \Gamma \}$$

- Multiply first eqn. by $v \in V$ & second eqn. by $q \in Q$
  & integrate over Ω

$$\begin{cases} \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega \nabla \cdot v \, p \, dx = \int_\Omega f v \, dx \\ \int_\Omega \nabla \cdot u \, q \, dx = 0 \end{cases}$$

Define bilinear forms
$$a : V \times V \to \mathbb{R} \quad \text{and} \quad b : V \times Q \to \mathbb{R}$$

$$\begin{cases} a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \\ b(v,w) = - \int_\Omega \nabla \cdot v \, w \, dx \end{cases}$$

<u>Weak formulation</u>: Find $(u,p) \in V \times Q$ s.t.

$$a(u,v) + b(v,p) = (f,v)$$
$$b(u,q) = 0$$

for all $(v,q) \in V \times Q$

$(\cdot,\cdot)$ scalar product in $L^2(\Omega)$

<u>(Mixed) Finite Element Method</u>

- Introduce finite element spaces $V_h \subset V$ and $Q_h \subset Q$
- Find $(u_h, p_h) \in V_h \times Q_h$ s.t.

$$a(u_h, v_h) + b(v_h, p_h) = (f, v_h)$$
$$b(u_h, q_h) = 0$$

for all $(v_h, q_h) \in V_h \times Q_h$

L1

$$V_h \subset H^1(\Omega) \qquad Q_h \subset Q$$

$$V_h = \mathrm{span}\{\varphi_1, \dots, \varphi_N\} \qquad Q_h = \mathrm{span}\{\psi_1, \dots, \psi_M\}$$

$$u_{kh}(x) = \sum_{j=1}^{N} u_{kj}\varphi_j(x)$$

$$p_h(x) = \sum_{j=1}^{M} p_j \psi_j(x)$$

$$\begin{cases} \displaystyle\sum_{k=1}^{n}\sum_{j=1}^{N} u_{kj}\int_\Omega \nabla\varphi_j(x)\cdot\nabla\varphi_i(x)\,dx + \sum_{k=1}^{n}\sum_{j=1}^{M} p_j \int_\Omega \psi_j \frac{\partial\varphi_i}{\partial x_k}\,dx = \sum_{k=1}^{n}\int_\Omega f_k\varphi_i\,dx & i=1,\dots,N \\ \displaystyle\sum_{k=1}^{n}\sum_{j=1}^{N} u_{kj}\int_\Omega \psi_i \frac{\partial\varphi_j}{\partial x_k} = 0 & i=1,\dots,M \end{cases}$$

$$\begin{array}{c c} \begin{array}{c} 2N \\ M \end{array} & \underbrace{\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix}}_{2N \quad M} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \end{array}$$

<u>Saddle-point problem</u>

Corresponds to a (constrained) minimization problem

Ladyzhenskaya-Babuška-Brezzi (LBB)

(inf-sup condition)

A finite element discretization of Stokes is stable if V_h & Q_h satisfy the LBB condition:

$$\min_{q_h \in Q_h} \max_{v_h \in V_h} \frac{(q_h, \nabla \cdot v_h)}{\|q_h\| \, \|\nabla v_h\|} \geq \gamma > 0$$

Ex. Crouzeix-Raviart

V_h $\quad$ Q_h

Ex. Taylor-Hood (P2/P1)

(L1)

(7)

A invertible $\Rightarrow$ $u = A^{-1}(f - Bp)$

$B^T u = 0 \Rightarrow \underbrace{B^T A^{-1} B p = B^T A^{-1} f}_{\text{Schur complement eqn.}}$

$\left(\begin{array}{l} S = B^T A^{-1} B \text{ symmetric} \\ \text{positive definite if } \ker(B) = \{0\} \end{array} \right)$

Schur complement methods

$$p_k = p_{k-1} - C^{-1}(BA^{-1}B^T p_{k-1} - BA^{-1} f)$$

C^{-1} preconditioner for $S = BA^{-1}B^T$

Usawa - algorithm: C^{-1} scaled identity matrix

$$\Rightarrow \begin{cases} 1. \text{ Solve } A u_k = f - B^T p_{k-1} \\ 2. \text{ Set } p_k = p_{k-1} + \alpha B u_k \end{cases}$$

Stabilized methods

Equal order interpolation is possible by stabilization of the standard Galerkin method.

Find $(u_h, p_h) \in V_h \times Q_h$ s.t.

$$a(u_h, v_h) + b(v_h, p_h) = (f, v_h)$$

$$b(u_h, q_h) + s(p_h, q_h) = 0$$

$$\begin{bmatrix} A & B \\ B^T & S \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

Ex. Brezzi - Pitkäranta $\nabla \cdot u = h^2 \Delta p$

$$\Rightarrow s(p_h, q_h) = \int_\Omega h^2 \nabla p_h \cdot \nabla q_h \, dx$$