F2

ACFM

Navier-Stokes equations

$$\begin{cases} \dot{u} + u \cdot \nabla u + \nabla p - Re^{-1} \Delta u = f \\ \nabla \cdot u = 0 \end{cases}$$

- $Re \to 0 \Rightarrow$ Stokes equations

$$\begin{cases} -\Delta u + \nabla p = f \\ \nabla \cdot u = 0 \end{cases}$$

- $Re \to \infty \Rightarrow$ Euler equations

$$\begin{cases} \dot{u} + u \cdot \nabla u + \nabla p = f \\ \nabla \cdot u = 0 \end{cases}$$

High Re transport dominated model problem

$$\begin{cases} \dot{u} + \beta \cdot \nabla u - \varepsilon \Delta u = f \\ \nabla \cdot \beta = 0 \end{cases} \qquad (\varepsilon > 0 \text{ small})$$

Ex $\begin{cases} -\varepsilon u'' + u' = 0 & \text{in } \Omega = (0,1) \\ u(0) = 1,\ u(1) = 0 \end{cases}$

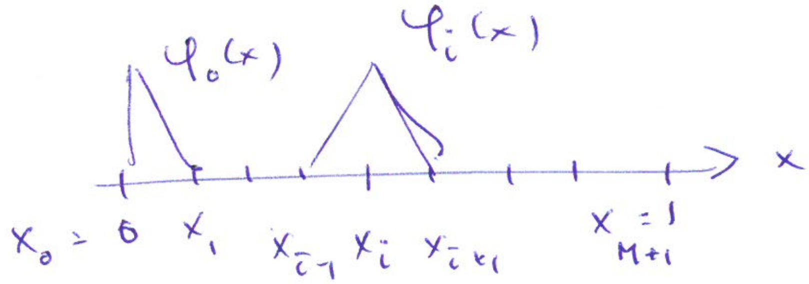
Galerkin FEM: Find $U \in V_h$ s.t.

$$\int_0^1 \varepsilon U' v' \, dx + \int_0^1 U' v \, dx = 0 \qquad \forall v \in V_h^0$$

$$V_h = \{ v \in H^1(0,1) : v(0) = 1,\ v(1) = 0 \}$$

$$V_h^0 = \{ v \in H^1(0,1) : v(0) = v(1) = 0 \}$$

$\mathcal{T}_h$: uniform mesh with M interior nodes



$h = \frac{1}{M+1}$

$$U(x) = \sum_{j=1}^{M} \xi_j \varphi_j(x) + u(0)\varphi_0(x) \longrightarrow \underset{M\times M}{A}\ \underset{M\times 1}{\xi} = \underset{M\times 1}{b}$$

$$\begin{cases} A_{ij} = \int_0^1 \varepsilon \varphi_j'(x)\varphi_i'(x)\,dx + \int_0^1 \varphi_j'(x)\varphi_i(x)\,dx \\ b_i = -\int_0^1 \varepsilon u(0)\varphi_0'(x)\varphi_i'(x)\,dx + \int_0^1 u(0)\varphi_0'(x)\varphi_i(x)\,dx = -\int_0^1 (\varepsilon\varphi_0'\varphi_i' + \varphi_0'\varphi_i)\,dx \end{cases}$$

$$A_{ii} = \int_{x_{i-1}}^{x_i} \left(\varepsilon \frac{1}{h}\frac{1}{h} + \frac{1}{h}\frac{x - x_{i-1}}{h} \right) dx + \int_{x_i}^{x_{i+1}} \varepsilon\left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right) + \left(-\frac{1}{h}\right)\left(\frac{x_{i+1}-x}{h}\right) dx = \frac{\varepsilon}{h} + \frac{1}{2} + \frac{\varepsilon}{h} - \frac{1}{2} = \frac{2\varepsilon}{h}$$

$$A_{i-1\,i} = \int_{x_{i-1}}^{x_i} \left(\varepsilon\left(\frac{-1}{h}\right)\frac{1}{h} + \left(\frac{-1}{h}\right)\frac{x - x_{i-1}}{h} \right) dx = -\frac{\varepsilon}{h} - \frac{1}{2}$$

(A non-symm.)

$$A_{i\,i+1} = \int_{x_i}^{x_{i+1}} \left(\varepsilon \frac{1}{h}\left(-\frac{1}{h}\right) + \frac{1}{h}\frac{x_{i+1} - x}{h} \right) dx = -\frac{\varepsilon}{h} + \frac{1}{2}$$

(F2)

Equation i:

$$\sum_{j=1}^{M} \xi_j A_{ij} = \xi_{i-1}\left(-\frac{\varepsilon}{h}-\frac{1}{2}\right) + \xi_i \frac{2\varepsilon}{h} + \xi_{i+1}\left(-\frac{\varepsilon}{h}+\frac{1}{2}\right) = 0$$

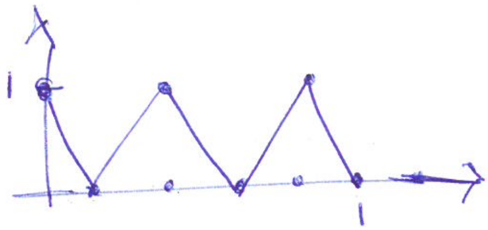
$(i>1)$

$\left(\frac{\varepsilon}{h}\right)$ large $\rightarrow -\xi_{i-1} + 2\xi_i - \xi_{i+1} = 0$ ("Poisson-like")

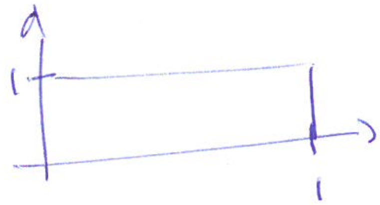
$\left(\frac{\varepsilon}{h}\right)$ small $\rightarrow -\frac{1}{2}\xi_{i-1} + \frac{1}{2}\xi_{i+1} = 0 \iff \xi_{i+1} = \xi_{i-1}$

$\Longrightarrow$ solution ξ oscillates (M even) or A singular (M odd)

M even: $\varepsilon = 0$

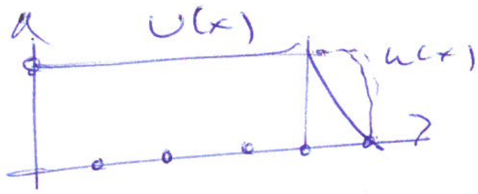


Exact solution:



Stabilization: add artificial viscosity $\varepsilon = \frac{h}{2}$

$\Rightarrow$ New equations: $\boxed{-\xi_{i-1} + \xi_i = 0}$ (upwind method)

$\Rightarrow$ 

- Galerkin FEM optimal for diffusion dominated problem
- —— " —— non-optimal for transport —— " ——
- For non-smooth exact solution: U contains spurious oscillations when using standard FEM when $\frac{\varepsilon}{h}$ small
- Artificial viscosity $\varepsilon \sim h \Rightarrow$ stability (no oscillations) but bad accuracy (no resolution of layers)

F2

Streamline diffusion stabilization

$$\begin{cases} \dot{u} + \beta \cdot \nabla u - \varepsilon \Delta u = 0 \\ \nabla \cdot \beta = 0 \end{cases}$$

$$(\dot{u}, v) + (\beta \cdot \nabla u, v) + (\delta(\beta \cdot \nabla u), \beta \cdot \nabla v) + (\varepsilon \nabla u, \nabla v) = 0$$

stabilization parameter $\delta \sim h$

Energy / stability estimate

Set $v = u \Rightarrow (\dot{u}, u) + (\beta \cdot \nabla u, u) + (\delta\, \beta \cdot \nabla u, \beta \cdot \nabla u)$
$+ (\varepsilon \nabla u, \nabla u) = 0$

$$\Rightarrow \quad \frac{d}{dt} \frac{1}{2} \|u\|^2 + \|\sqrt{\delta}\, \beta \cdot \nabla u\|^2 + \|\sqrt{\varepsilon} \nabla u\|^2 = 0$$

$$(\beta \cdot \nabla u, u) = (-(\cancel{\nabla \cdot \beta})^{0} u - \beta \cdot \nabla u, u) = -(\beta \cdot \nabla u, u)$$

$$\Rightarrow \quad (\beta \cdot \nabla u, u) = 0$$

$$(\dot{u}, u) = \int_\Omega \dot{u} u \, dx = \frac{d}{dt} \frac{1}{2} \int_\Omega u^2 dx = \frac{d}{dt} \frac{1}{2} \|u\|^2$$

$$\Rightarrow \quad \|u(T)\|^2 + 2 \int_0^T \|\sqrt{\delta}\, \beta \cdot \nabla u\|^2 dt$$
$$+ 2 \int_0^T \|\sqrt{\varepsilon} \nabla u\|^2 dt = \underbrace{\|u(0)\|^2}_{C > 0}$$

F2

Streamline diffusion stabilization NSE

Find $(U,P) \in V_h \times Q_h$ s.t.

$$(\dot{U},v) + (U\cdot\nabla U, v) + (\nu\nabla U, \nabla v) - (P, \nabla\cdot v) + (\nabla\cdot U, q)$$
$$+ (\delta_1\, U\cdot\nabla U, U\cdot\nabla v) + (\delta_2 \nabla P, \nabla q) = 0$$
$$\forall (v,q) \in V_h \times Q_h$$

Stability estimate : $(v,q) = (U,P) \Rightarrow$

$$\frac{d}{dt}\frac{1}{2}\|U\|^2 + \|\sqrt{\nu}\,\nabla U\|^2 + \|\sqrt{\delta_1}\,U\cdot\nabla U\|^2 + \|\sqrt{\delta_2}\,\nabla P\|^2 = 0$$

Galerkin Least Squares Stabilization (GLS)

$$(\dot{U},v) + (U\cdot\nabla U, v) + (\nu\nabla U, \nabla v) - (P, \nabla\cdot v) + (\nabla\cdot U, q)$$
$$+ (\delta_1(\dot{U} + U\cdot\nabla U + \nabla P), \dot{v} + U\cdot\nabla v + \nabla q)$$
$$+ (\delta_2(\nabla\cdot U), \nabla\cdot v) = 0 \quad \forall (v,q) \in V_h \times Q_h$$

F2

Semidiscretization by the θ-method

Discretize in space by FEM, then discretize the system of ODEs with the θ-method.

Ex. Streamline diffusion method

For all time step intervals $I_n = (t_n, t_{n+1})$ find $(U^{n+1}, P^{n+1}) \in V_h \times Q_h$ s.t.

$$
\begin{aligned}
&\left(\frac{U^{n+1}-U^n}{k}, v\right) + \left(U^{n+\theta}\cdot\nabla U^{n+\theta}, v\right) - \left(P^{n+\theta}, \nabla\cdot v\right) \\
&+ \left(\nabla\cdot U^{n+\theta}, q\right) + \left(\nu\nabla U^{n+\theta}, \nabla v\right) \\
&+ \left(\delta_1\left(U^{n+\theta}\cdot\nabla U^{n+\theta}\right), U^{n+\theta}\cdot\nabla v\right) \\
&+ \left(\delta_2\nabla P^{n+\theta}, \nabla q\right) = 0 \qquad \forall (v,q)\in V_h\times Q_h
\end{aligned}
$$