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ACFM

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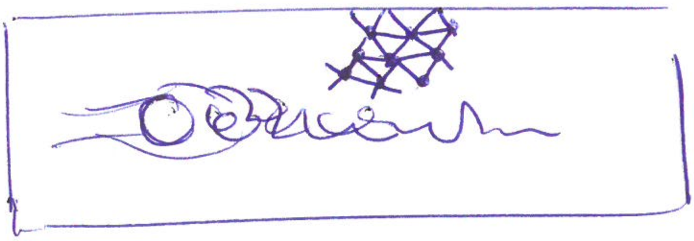
Adaptive methods



- What is the error?
- How can we minimize the error?
- Can we optimize our method?

For a general PDE: $A(u) = f$

- the solution is unknown
- the error $e = u - U$ is unknown
- the residual $R(U) = f - A(U)$ is computable

□ How can we relate the unknown error e to the computable residual $R(U)$?

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A posteriori error estimation

- Error estimates in terms of the computed approximate solution U.

- We can measure the error e directly, e.g. $\|e\|$, or the error in some functional of the solution $|M(u) - M(U)|$

- All linear functionals can be expressed in terms of an inner product: (Riesz representation theorem)

$M(w) = (w, \psi)$ (ψ the Riesz representer)

<u>Ex.</u> $\|e\|^2 = (e, e) = M(e)$ (with $\psi = e$)

$$\frac{1}{|\omega|} \int_\omega e \, dx = \frac{1}{|\omega|} \int_\Omega e \, \chi_\omega \, dx = M(e) \qquad \left(\psi = \frac{\chi_\omega}{|\omega|} \right)$$

$(\omega \subset \Omega \subset \mathbb{R}^n)$

$$e(z) = \int_\Omega e(x) \, \delta_z(x) \, dx = M(e) \qquad (\psi = \delta_z)$$

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Adjoint operator

$A : X \to Y$ bounded linear operator

$A^* : Y^* \to X^*$ dual/adjoint operator

Def. $(A(v), w) = (v, A^*(w^*))$ $\begin{cases} v \in X \\ w^* \in Y^* \end{cases}$

Ex. $A(v) = -\Delta v(x) \qquad x \in \Omega \subset R^n$

$v \in H_0^1(\Omega) = X, \; w^* \in H_0^1(\Omega) = Y^*$

$(A(v), w^*) = (-\Delta v, w^*) = (\nabla v, \nabla w^*) - (\nabla v \cdot n, w^*)_\Gamma$

$= (v, -\Delta w^*) - (\nabla v \cdot n, w^*)_\Gamma + (v, \nabla w^* \cdot n)_\Gamma$

A self-adjoint

$= (v, -\Delta w^*) = (v, A^*(w^*))$

Ex. $A(v) = -v''(x) \qquad x \in (0,1)$

$v \in X = \{v \in H^1(0,1), v(0) = v'(0) = 0\}$

$(A(v), w^*) = (-v'', w^*) = \int_0^1 (-v'') w^* \, dx$

$= \int_0^1 v' w^{*\prime} \, dx - \left[v' w^* \right]_0^1$

$= \int_0^1 v(-w^{*\prime\prime}) \, dx - [v' w]_0^1 + [v w^{*\prime}]_0^1 = (v, A^*(w^*))$

$A^*(w^*) = -w^{*\prime\prime}(x), \quad Y^* = \{w^* \in H^1(0,1), w^*(1) = w^{*\prime}(1) = 0\}$

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<u>Ex.</u> $A(v) = \beta \cdot \nabla v$

$$\mathbb{X} = \cancel{H_0^1(\Omega)} \{ v \in H^1(\Omega),\ v|_{\Gamma_+} = 0 \}$$

$$(A(v), w^*) = (\beta \cdot \nabla v, w^*) =$$

$$= (v, -\nabla(\beta w^*)) + \underbrace{\cancel{(\ ,\ )}}_{(v, (\beta \cdot n) w^*)_\Gamma}$$

$$= (v, A^*(w^*)) \qquad \Gamma = \Gamma_+ \cup \Gamma_-$$

$$A^*(w^*) = -\nabla(\beta w^*), \quad \mathbb{Y}^* = \{ w^* \in H^1(\Omega), w^*|_{\Gamma_-} = 0 \}$$

<u>Ex.</u> $A(v) = \dot{v} + \beta \cdot \nabla v$

$$\mathbb{X} = \{ v \in L_2(I; H^1(\Omega)), \dot{v} \in L_2(I; L_2(\Omega)) \}$$
$$v(0) = 0,\ v|_{\Gamma_+} = 0$$

$$(A(v), w^*) = \int_0^T \int_\Omega (\dot{v} + \beta \cdot \nabla v) w^* \, dx \, dt$$

$$= \int_0^T \int_\Omega v(-\dot{w}^*) \, dx \, dt + \left[\int_\Omega v w^* \, dx \right]_0^T$$

$$+ \int_0^T \int_\Omega v(-\nabla \cdot (\beta w^*)) \, dx \, dt + \int_0^T (v, (\beta \cdot n) w^*)_\Gamma \, dt$$

$$= (v, A^*(w^*))$$

$$A^*(w^*) = -\dot{w}^* - \nabla \cdot (\beta w^*)$$

$$\mathbb{Y}^* = \{ w^* \in L_2(I; H^1(\Omega)), \dot{w}^* \in L_2(I; L_2(\Omega)), w^*|_{\Gamma_-} = 0, \ w^*(T) = 0 \}$$

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The adjoint problem

For PDE $\begin{cases} A(u) = f \\ u \in X \end{cases}$ ($A(\cdot)$ linear)

we def. adjoint problem

$$\begin{cases} A^*(\varphi) = \psi \\ \varphi \in Y^* \end{cases}$$

to estimate error $M(u) - M(U)$

with $M(u) = (u, \psi)$

(ψ Riesz representer of $M(\cdot)$)

(□ With U a Galerkin FEM ~~solution~~

we get: Find $U \in V_h \subset V$ s.t.

$$(A(U), v) = (f, v) \quad \forall v \in V_h$$)

$$\underline{M(u) - M(U)} = M(e) = (e, \psi)$$

$$= (e, A^*(\varphi)) = (A(e), \varphi) = \underline{(R(U), \varphi)}$$

$$A(e) = A(u - U) = A(u) - A(U) = f - A(U)$$

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- For U Galerkin FEM solution:

Find $U \in V_h$ s.t. $(A(U), v) = (f, v)$ $\forall v \in V_h$

$$\Rightarrow M(u) - M(U) = (R(U), \varphi) = (R(U), \varphi - \pi_h \varphi)$$

$\pi_h \varphi \in V_h$

Cauchy–Schwarz ineq.

$$\leq \| h^2 R(U) \| \underbrace{\| h^{-2} (\varphi - \pi_h \varphi) \|}_{\leq C_i \| D^2 \varphi \|}$$

Interpolation ~~est~~ estimate

- Global a post. error estimate

$$M(u) - M(U) \leq \| h^2 R(U) \| \, C_i \| D^2 \varphi \|$$

- Local over all cells K in mesh $\mathcal{T}^h$

$$M(u) - M(U) \leq \sum_{K \in \mathcal{T}_h} \| h^2 R(U) \|_K \, C_i \| D^2 \varphi \|_K$$

"Do-nothing" estimate

$$M(u) - M(U) = (R(U), \varphi) \approx (R(U), \varphi_h)$$

$$= \sum_{K \in \mathcal{T}_h} (R(U), \varphi)_K \approx \sum_{K \in \mathcal{T}_h} (R(U), \varphi_h)_K$$

($\varphi_h \approx \varphi$ numerical approximation)

If $\varphi_h \in V_h \Rightarrow \underline{(R(U), \varphi_h) = 0}$

Galerkin FEM

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Navier–Stokes equation

$$\begin{cases} \dot{u} + u\cdot\nabla u + \nabla p - \nu \Delta u = f \\ \nabla\cdot u = 0 \\ u|_\Gamma = 0 \ , \ u(x,0) = u^0(x) = 0 \end{cases}$$

$$A(\hat{u}) = A((u,p)) = \begin{pmatrix} \dot{u} + u\cdot\nabla u + \nabla p - \nu\Delta u \\ \nabla\cdot u \end{pmatrix}$$

$$X = \{(u,p) \in V \times Q : u|_\Gamma = 0, \ u(x,0) = 0\}$$

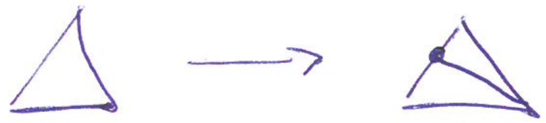
Adjoint Navier–Stokes eqns.

$$\begin{cases} -\dot{\varphi} - u\cdot\nabla\varphi + \nabla U^T\cdot\varphi + \nabla\theta - \nu\Delta\varphi = \psi_1 \\ \nabla\cdot\varphi = \psi_2 \\ \varphi|_\Gamma = 0, \ \varphi(x,T) = 0 \end{cases}$$

$$A^*(\hat{\varphi}) = \begin{pmatrix} -\dot{\varphi} - u\cdot\nabla\varphi + \nabla U^T\cdot\varphi + \nabla\theta - \nu\Delta\varphi \\ \nabla\cdot\varphi \end{pmatrix}$$

$$Y^* = \{(\varphi,\theta) \in V \times Q : \varphi|_\Gamma = 0, \ \varphi(x,T) = 0\}$$

Mesh refinement



bisection (recursive)

red-green refinement

$$M(u) - M(U) = (R(U), \varphi) = \sum_{K \in T_h} \mathcal{E}_K$$

$$\mathcal{E}_K = (R(U), \varphi)_K \quad \text{(error indicator)}$$

$$\sum_{K \in T_h} \mathcal{E}_K < TOL \quad \text{(stopping criterion)}$$