Namn: .... Personnummer: ....

Lappskrivning 2: Lösningsförslag

Måndag 25 april 2016 8:15-9:45
Differential- och integralkalkyl II, del 2, SF1603, Flervariabelanalys
Inga hjälpmedel är tillåtna.
Max: 12 poäng

Version A

1. (4 poäng) Beräkna volymen av kroppen $K \subset \mathbb{R}^3$ som begränsas av de tre planen

$$y = 0, \qquad z = 0, \qquad z = 3 - x + y,$$

samt av den paraboliska cylindern $y = 1 - x^2$.

Lösning: Låt $D \subset \mathbb{R}^2$ vara projektionen av K på xy-planet. Då D begränsas av $y = 0$ och $y = 1 - x^2$ har vi

$$D = \{(x, y) \in \mathbb{R}^2 \,|\, -1 \leq x \leq 1,\ 0 \leq y \leq 1 - x^2\}.$$

Kroppen K består av volymen över D som begränsas uppåt av planet $z = 3-x+y$ och nedåt av planet $z = 0$ (notera att $3 - x + y \geq 0$ för $(x, y) \in D$). Vi finner

$$\begin{aligned}
\text{Volym}(K) &= \iint_D (3 - x + y - 0)\,dxdy \\
&= \int_{-1}^{1} \int_0^{1-x^2} (3 - x + y)\,dydx \\
&= \int_{-1}^{1} \left[(3 - x)y + \frac{y^2}{2}\right]_{y=0}^{1-x^2} dx \\
&= \int_{-1}^{1} \left((3 - x)(1 - x^2) + \frac{(1 - x^2)^2}{2}\right) dx \\
&= \int_{-1}^{1} \left(\frac{7}{2} - x - 4x^2 + x^3 + \frac{x^4}{2}\right) dx \\
&= 2\int_0^1 \left(\frac{7}{2} - 4x^2 + \frac{x^4}{2}\right) dx \\
&= 2\left[\frac{7x}{2} - 4\frac{x^3}{3} + \frac{x^5}{10}\right]_{x=0}^{1} \\
&= 2\left(\frac{7}{2} - \frac{4}{3} + \frac{1}{10}\right) \\
&= \frac{68}{15}.
\end{aligned}$$

Svar: $\text{Volym}(K) = \frac{68}{15}$.

2. (4 poäng) Beräkna $\iint_Y f(x,y,z)dS$ där $f(x,y,z) = y^2\sqrt{x}$ och ytan Y har parameterframställningen

$$\mathbf{r}(u,v) = (v^2, u, u+v), \qquad 0 \le u \le 1, \quad 0 \le v \le 1.$$

Lösning: Integralen ges av

$$\iint_Y f(x,y,z)dS = \int_0^1 \int_0^1 f(\mathbf{r}(u,v))|\mathbf{r}'_u \times \mathbf{r}'_v| dv du.$$

Eftersom

$$\mathbf{r}'_u \times \mathbf{r}'_v = (0,1,1) \times (2v,0,1) = (1, 2v, -2v),$$

får vi

$$|\mathbf{r}'_u \times \mathbf{r}'_v| = \sqrt{1 + (2v)^2 + (-2v)^2} = \sqrt{1+8v^2}.$$

Det följer att

$$\begin{aligned}\iint_Y f(x,y,z)dS &= \int_0^1 \int_0^1 u^2 v\sqrt{1+8v^2} dv du \\ &= \left(\int_0^1 u^2 du\right)\left(\int_0^1 v\sqrt{1+8v^2}dv\right) \\ &= \frac{1}{3}\left[\frac{(1+8v^2)^{3/2}}{24}\right]_0^1 \\ &= \frac{1}{3}\cdot\frac{9^{3/2}}{24} = \frac{1}{3}\cdot\frac{3^3}{24} = \frac{13}{36}.\end{aligned}$$

Svar: $\iint_Y f(x,y,z)dS = \frac{13}{36}$.

3. (4 poäng) Bestäm ett uttryck för ekvationen

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 4\frac{\partial^4 u}{\partial z^4} = 0$$

i cylindriska koordinater.

Lösning: De cylindriska koordinaterna (r, φ, z) uppfyller

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad z = z.$$

Alltså är

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix},$$

vilket ger

$$\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix}^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\frac{1}{r}\sin\varphi & \frac{1}{r}\cos\varphi \end{pmatrix}.$$

Det följer att

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} = \cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial \varphi}$$

och

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi} = \sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial}{\partial \varphi}.$$

Detta ger

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} =& \left(\cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial \varphi}\right)\left(\cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial \varphi}\right)u \\
&+ \left(\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial}{\partial \varphi}\right)\left(\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial}{\partial \varphi}\right)u \\
=& \left(\cos^2\varphi\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos\varphi\sin\varphi\frac{\partial}{\partial \varphi} - \frac{1}{r}\cos\varphi\sin\varphi\frac{\partial^2}{\partial r\partial \varphi}\right. \\
&\left. + \frac{1}{r}\sin^2\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\partial \varphi} + \frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial \varphi} + \frac{1}{r^2}\sin^2\varphi\frac{\partial^2}{\partial \varphi^2}\right)u \\
&+ \left(\sin^2\varphi\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial \varphi} + \frac{1}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\partial \varphi}\right. \\
&\left. + \frac{1}{r}\cos^2\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\sin\varphi\frac{\partial^2}{\partial r\partial \varphi} - \frac{1}{r^2}\cos\varphi\sin\varphi\frac{\partial}{\partial \varphi} + \frac{1}{r^2}\cos^2\varphi\frac{\partial^2}{\partial \varphi^2}\right)u.
\end{aligned}$$

Förenkling ger

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)u.$$

Svar: Ekvationen ges i cylindriska koordinater av

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + 4\frac{\partial^4 u}{\partial z^4} = 0.$$

Version B

1. (4 poäng) Beräkna volymen av kroppen $K \subset \mathbb{R}^3$ som begränsas av de tre planen

$$y = 0, \qquad z = 0, \qquad z = 4 - x + y,$$

samt av den paraboliska cylindern $y = 1 - x^2$.

Lösning: Låt $D \subset \mathbb{R}^2$ vara projektionen av K på xy-planet. Då D begränsas av $y = 0$ och $y = 1 - x^2$ har vi

$$D = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1,\ 0 \leq y \leq 1 - x^2\}.$$

Kroppen K består av volymen över D som begränsas uppåt av planet $z = 4 - x + y$

och nedåt av planet $z = 0$ (notera att $4 - x + y \geq 0$ för $(x, y) \in D$). Vi finner

$$\begin{aligned}
\text{Volym}(K) &= \iint_D (4 - x + y - 0)dxdy \\
&= \int_{-1}^{1} \int_0^{1-x^2} (4 - x + y)dydx \\
&= \int_{-1}^{1} \left[(4 - x)y + \frac{y^2}{2}\right]_{y=0}^{1-x^2} dx \\
&= \int_{-1}^{1} \left((4 - x)(1 - x^2) + \frac{(1 - x^2)^2}{2}\right)dx \\
&= \int_{-1}^{1} \left(\frac{9}{2} - x - 5x^2 + x^3 + \frac{x^4}{2}\right)dx \\
&= 2\int_0^1 \left(\frac{9}{2} - 5x^2 + \frac{x^4}{2}\right)dx \\
&= 2\left[\frac{9x}{2} - 5\frac{x^3}{3} + \frac{x^5}{10}\right]_{x=0}^{1} \\
&= 2\left(\frac{9}{2} - \frac{5}{3} + \frac{1}{10}\right) \\
&= \frac{88}{15}.
\end{aligned}$$

Svar: Volym$(K) = \frac{88}{15}$.

2. (4 poäng) Beräkna $\iint_Y f(x, y, z)dS$ där $f(x, y, z) = y\sqrt{x}$ och ytan Y har parameterframställningen

$$\mathbf{r}(u, v) = (v^2, u, u + v), \qquad 0 \leq u \leq 1, \quad 0 \leq v \leq 1.$$

Lösning: Integralen ges av

$$\iint_Y f(x, y, z)dS = \int_0^1 \int_0^1 f(\mathbf{r}(u, v))|\mathbf{r}'_u \times \mathbf{r}'_v|dvdu.$$

Eftersom

$$\mathbf{r}'_u \times \mathbf{r}'_v = (0, 1, 1) \times (2v, 0, 1) = (1, 2v, -2v),$$

får vi

$$|\mathbf{r}'_u \times \mathbf{r}'_v| = \sqrt{1 + (2v)^2 + (-2v)^2} = \sqrt{1 + 8v^2}.$$

Det följer att

$$\begin{aligned}
\iint_Y f(x, y, z)dS &= \int_0^1 \int_0^1 uv\sqrt{1 + 8v^2}dvdu \\
&= \left(\int_0^1 udu\right)\left(\int_0^1 v\sqrt{1 + 8v^2}dv\right) \\
&= \frac{1}{2}\left[\frac{(1 + 8v^2)^{3/2}}{24}\right]_0^1 \\
&= \frac{1}{2} \cdot \frac{9^{3/2}}{24} = \frac{1}{2} \cdot \frac{3^3}{24} = \frac{13}{24}.
\end{aligned}$$

Svar: $\iint_Y f(x,y,z)dS = \frac{13}{24}$.

3. (4 poäng) Bestäm ett uttryck för ekvationen

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 5\frac{\partial^3 u}{\partial z^3} = 0$$

i cylindriska koordinater.

Lösning: De cylindriska koordinaterna (r, φ, z) uppfyller

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad z = z.$$

Alltså är

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix},$$

vilket ger

$$\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix}^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\frac{1}{r}\sin\varphi & \frac{1}{r}\cos\varphi \end{pmatrix}.$$

Det följer att

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi} = \cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial \varphi}$$

och

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi} = \sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial}{\partial \varphi}.$$

Detta ger

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} =& \left(\cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial \varphi}\right)\left(\cos\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial \varphi}\right)u \\
&+ \left(\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial}{\partial \varphi}\right)\left(\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial}{\partial \varphi}\right)u \\
=& \left(\cos^2\varphi\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos\varphi\sin\varphi\frac{\partial}{\partial \varphi} - \frac{1}{r}\cos\varphi\sin\varphi\frac{\partial^2}{\partial r\partial \varphi}\right. \\
&\left. + \frac{1}{r}\sin^2\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\partial\varphi} + \frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{r^2}\sin^2\varphi\frac{\partial^2}{\partial\varphi^2}\right)u \\
&+ \left(\sin^2\varphi\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\partial\varphi}\right. \\
&\left. + \frac{1}{r}\cos^2\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\sin\varphi\frac{\partial^2}{\partial r\partial\varphi} - \frac{1}{r^2}\cos\varphi\sin\varphi\frac{\partial}{\partial\varphi} + \frac{1}{r^2}\cos^2\varphi\frac{\partial^2}{\partial\varphi^2}\right)u.
\end{aligned}$$

Förenkling ger

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)u.$$

Svar: Ekvationen ges i cylindriska koordinater av

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2} - 5\frac{\partial^3 u}{\partial z^3} = 0.$$