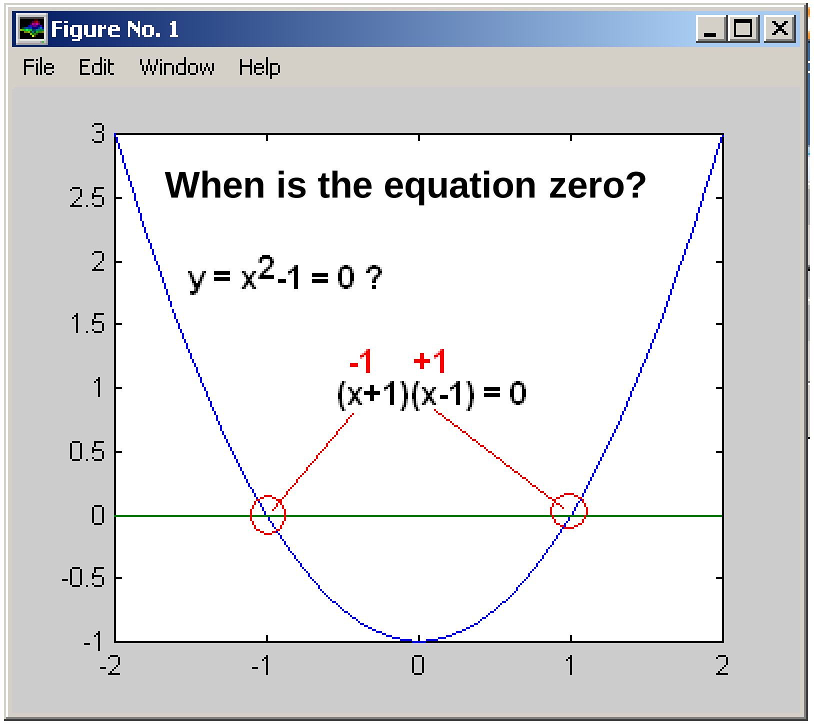
Complex numbers refresher

How many solutions have a quadratic equation?

$$y = x^2 - 1 = 0$$



Two solutions!

Do you remember the difference of two squares?

$$y = (x+1)(x-1) = x^2 + x - x - 1^2 = x^2 - 1$$

$$y = 0 \quad \Leftrightarrow \quad (x+1)(x-1) = 0$$

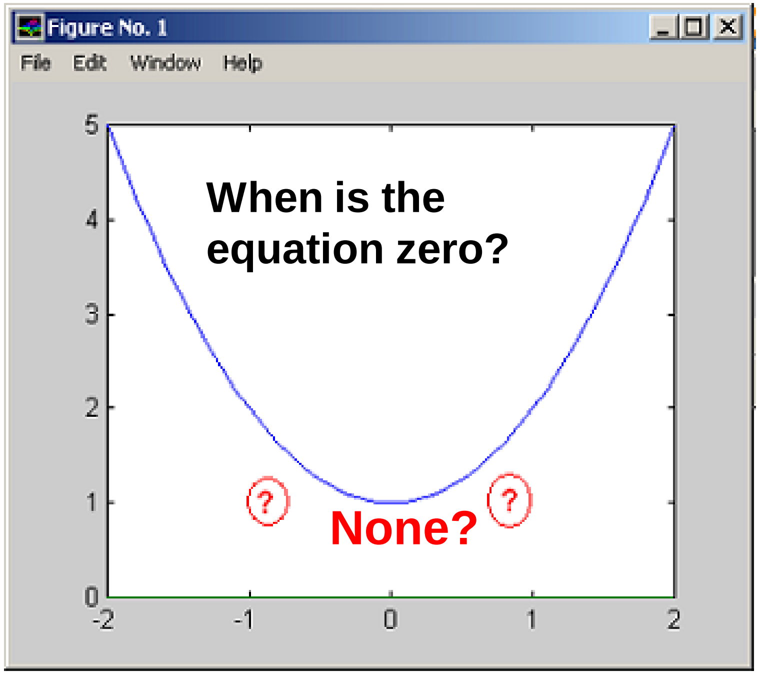
$$x_1 = -1 \quad x_2 = +1$$

The answer might as well be written:

$$x_1 = -\sqrt{1} \quad x_2 = +\sqrt{1}$$

How many solutions has this?

$y = x^2 + 1 = 0$



Two solutions!

If it were the case that the number "root of minus one" would exist!

$$y = (x + \sqrt{-1})(x - \sqrt{-1}) = x^2 + \sqrt{-1}x - \sqrt{-1}x - \underbrace{\sqrt{-1}^2}_{+1} = x^2 + 1$$

$$y = 0 \quad \Leftrightarrow \quad (x + \sqrt{-1})(x - \sqrt{-1}) = 0$$

$$x_1 = -\sqrt{-1} \quad x_2 = +\sqrt{-1}$$

Imaginary numbers

In mathematics you want there to be as many solutions to an equation as the equation degrees.

This will be the case if we introduce "the square root of minus one" as a number. This number is usually referred to as **i** (imaginary unit), or **j** because the main use of imaginary numbers are in electricity, where the letter "i" is already taken to denote current.

In addition to our usual real dimension with **1** ($\sqrt{1} = 1$) as unit we introduce an extra imaginary dimension with **j** ($\sqrt{-1}$) as the unit.

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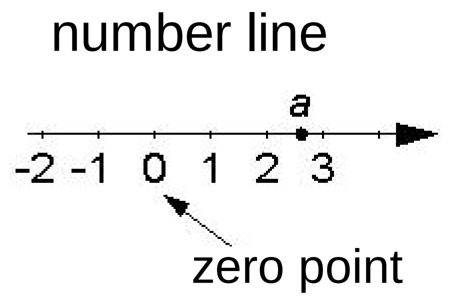
Number line

Would you like to "rehearse" Your knowledge of complex numbers, so go ahead and continue reading here ...



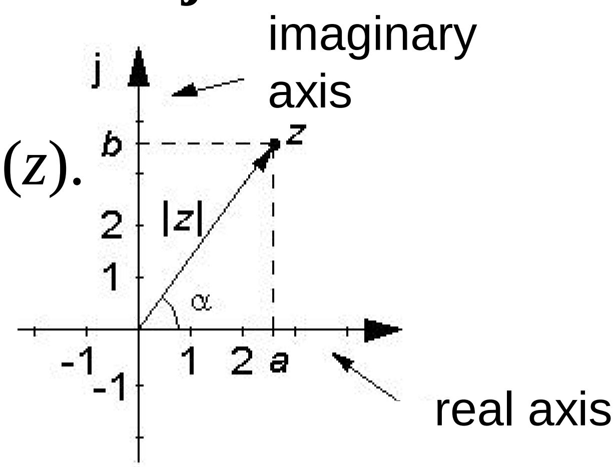
A common, real numbers is usually illustrated as a point on the so-called number line. The number amount is represented by the distance from the point in question to the number line zero.

The complex plane

$$z = a \cdot \sqrt{1} + b \cdot \sqrt{-1} = a + \mathrm{j}b$$

A complex number *z* contains two components. It can be written *a* + **j***b*. Here ***a*** and ***b*** are real numbers. **j** is ”$\sqrt{-1}$ ” and is called the imaginary unit.

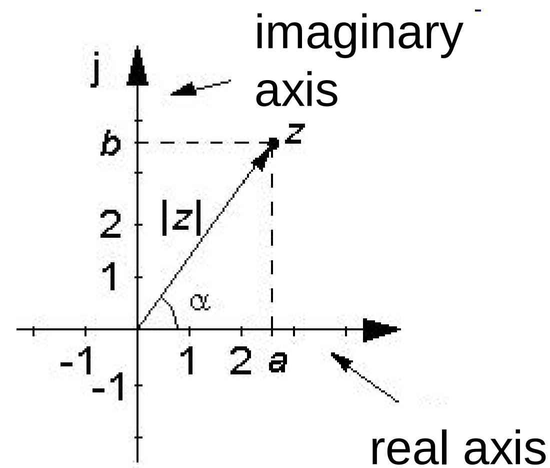
a is the complex number **real part**, Re(*z*).
b is it’s **imaginary part**, Im(*z*).



Every complex number can be represented as a point in a two-dimensional coordinate system, **the complex plane**.

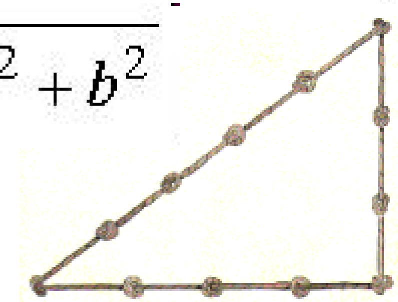
Amount and Argument

A complex number "coordinates" can alternatively be expressed polar, as the number **amount** | z |, the distance from the origin, and the **argument** α, the angle to the real axis.



The amount is calculated by using the **Pythagorean theorem**.

$$|z| = \sqrt{a^2 + b^2}$$



Right angle, rope with knots, found in a pyramid

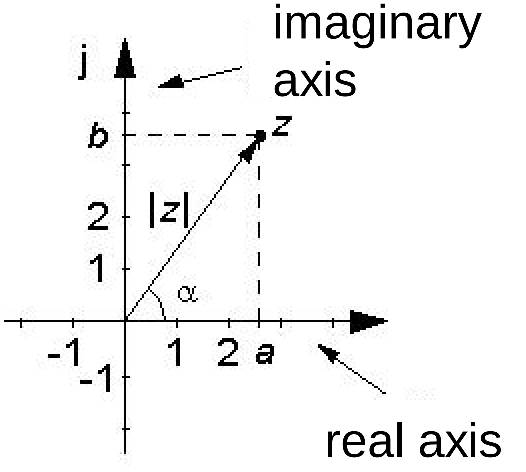
To calculate the argument you need to use trigonometric functions.

$$\tan(\alpha) = \frac{b}{a}$$

Amount and Argument

$$a = |z|\cos(\alpha) \quad b = |z|\sin(\alpha)$$

$$z = |z|(\cos(\alpha) + j \cdot \sin(\alpha))$$

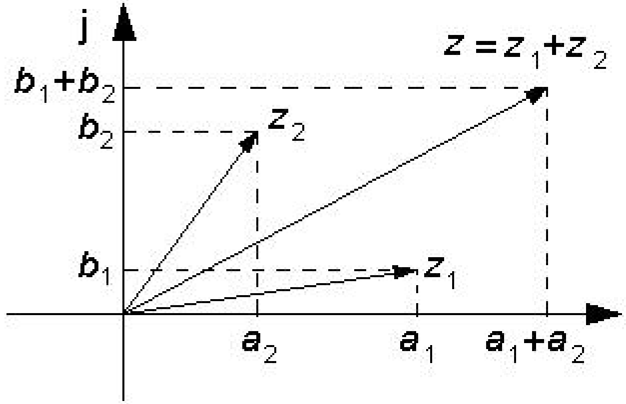


$$|z| = \sqrt{[\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2}$$

$$\arg(z) = \alpha = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right) + n \cdot 2\pi \qquad a > 0$$

$$\arg(z) = \alpha = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right) + \pi + n \cdot 2\pi \qquad a < 0$$

Addition of complex numbers



$$z = z_1 + z_2 = \mathrm{Re}(z_1) + \mathrm{Re}(z_2) + \mathrm{j}\left(\mathrm{Im}(z_1) + \mathrm{Im}(z_2)\right)$$

$$z_1 = a_1 + \mathrm{j}b_1 \qquad z_2 = a_2 + \mathrm{j}b_2$$

$$z = z_1 + z_2 = \left(a_1 + a_2\right) + \mathrm{j}\left(b_1 + b_2\right) = \mathrm{Re}(z) + \mathrm{j} \cdot \mathrm{Im}(z)$$

The figure shows what the addition means in the complex plane. The pointer z becomes equal to the geometrical sum of the pointers for z_1 and z_2.

Subtraction

$$z = z_1 - z_2 = \mathrm{Re}(z_1) - \mathrm{Re}(z_2) + \mathrm{j}\big(\mathrm{Im}(z_1) - \mathrm{Im}(z_2)\big)$$

In plane becomes pointer for z equal to the geometric difference between the pointers for z_1 and z_2.

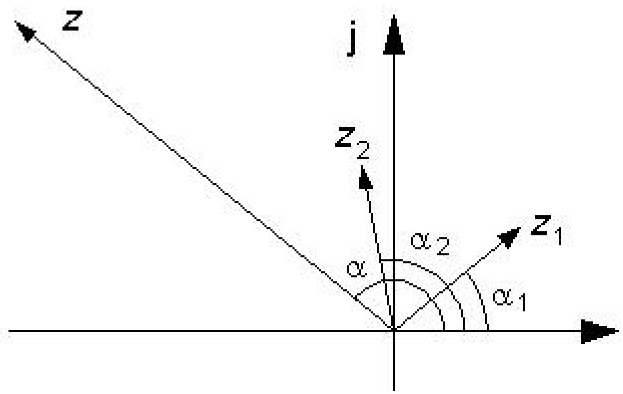
Multiplication

Multiplication rule is most easily demonstrated with an example.

$$z_1 = a_1 + \mathrm{j}b_1 \qquad z_2 = a_2 + \mathrm{j}b_2 \qquad (\mathrm{j})^2 = -1$$

$$z = z_1 \cdot z_2 = (a_1 + \mathrm{j}b_1)(a_2 + \mathrm{j}b_2) = (a_1a_2 - b_1b_2) + \mathrm{j}(a_1b_2 + a_2b_1)$$

Multiplication in polar form



Multiplicate the amounts and add the arguments!

$$z = z_1 \cdot z_2$$
$$= |z_1|\left(\cos(\alpha_1) + \mathrm{j}\sin(\alpha_1)\right) \cdot |z_2|\left(\cos(\alpha_2) + \mathrm{j}\sin(\alpha_2)\right)$$
$$= |z_1| \cdot |z_1|\left(\cos(\alpha_1 + \alpha_2) + \mathrm{j}\sin(\alpha_1 + \alpha_2)\right)$$

Which means that

$$|z| = |z_1 \cdot z_2| = |z_1| \cdot |z_2| \qquad \arg(z) = \arg(z_1) + \arg(z_2)$$

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Division

$$z_1 = a_1 + \mathrm{j}b_1 \qquad z_2 = a_2 + \mathrm{j}b_2$$

$$z = \frac{z_1}{z_2} = \frac{a_1 + \mathrm{j}b_1}{a_2 + \mathrm{j}b_2}$$

Now you often want to have the results in the form a+jb and if so, we extends the denominator with "conjugates quantity" a_2 - jb_2.

$$z = \frac{(a_1 + \mathrm{j}b_1)(a_2 - \mathrm{j}b_2)}{a_2^2 + b_2^2} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \mathrm{j}\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$$

Division in polar form

If the numbers are in polar form, the division rule looks like this:

$$z = \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\left(\cos(\alpha_1 - \alpha_2) - \mathrm{j}\sin(\alpha_1 - \alpha_2)\right)$$

$$|z| = \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \arg(z) = \arg(z_1) - \arg(z_2)$$

Divide amounts and subtract arguments!

Summary

$$\left|z_1 \cdot z_2\right| = \left|z_1\right| \cdot \left|z_2\right|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{\left|z_1\right|}{\left|z_2\right|}$$

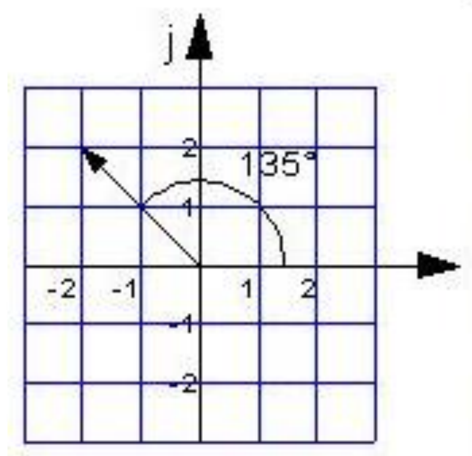
$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

Exercises

Question

In which direction are the pointer pointing $z = -2 + j2$?

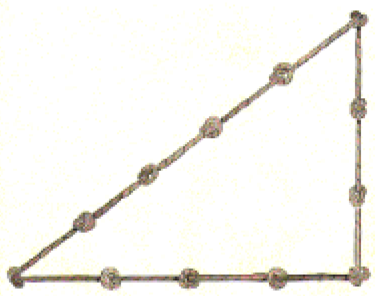


Exercises

Question

How long is the pointer $z = 3 + j4$?

$$\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$



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Exercises

$$z = z_1 \cdot z_2$$

Question

$z_1 = \mathrm{j}$ and $z_2 = -1 - \mathrm{j}$

Derive $|z|$ and $\arg(z)$ for $z = z_1 \cdot z_2$? *Algebraically* :

$$z = z_1 \cdot z_2 = \mathrm{j} \cdot (-1 - \mathrm{j}) = -\mathrm{j} - \mathrm{j}^2 = 1 - \mathrm{j}$$

$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\arg(z) = \alpha = \arctan\left(\frac{-1}{1}\right) = \arctan(-1) =$$

$$-\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$$

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Exercises

$z = z_1 \cdot z_2$

Question

z_1 = j and z_2 = -1 – j

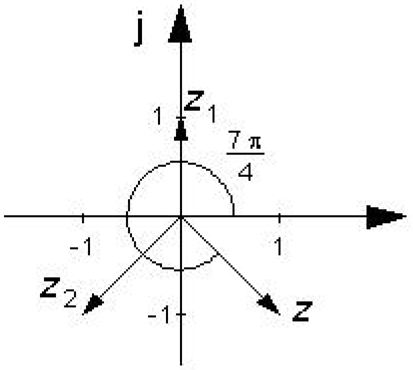
Derive $|z|$ and $\arg(z)$ for $z = z_1 \cdot z_2$? *Polar*:

$$|z_1| = 1 \qquad |z_2| = \sqrt{2}$$

$$z_1 = 1\left(\cos\frac{\pi}{2} + \mathrm{j}\sin\frac{\pi}{2}\right) \quad z_2 = \sqrt{2}\left(\cos\frac{5\pi}{4} + \mathrm{j}\sin\frac{5\pi}{4}\right)$$

$$z = z_1 \cdot z_2 = 1 \cdot \sqrt{2}\left(\cos\left(\frac{\pi}{2} + \frac{5\pi}{4}\right) + \mathrm{j}\sin\left(\frac{\pi}{2} + \frac{5\pi}{4}\right)\right)$$

$$|z| = \sqrt{2} \qquad \arg(z) = \frac{7\pi}{4}$$



Multiplication with* j *apparently means turning 90° !

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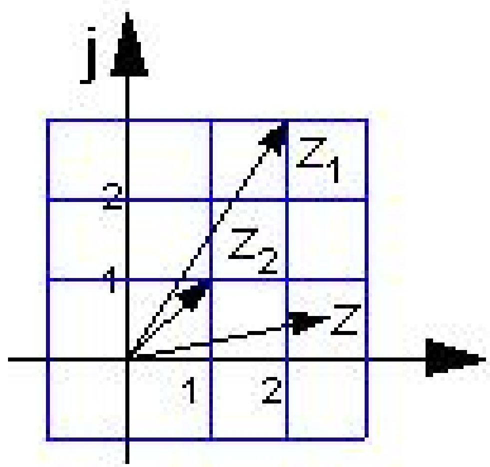
Exercises

$$z = \frac{z_1}{z_2}$$

Question

$z_1 = 2 + 3\text{j}$ and $z_2 = 1 + \text{j}$

Derive $z = z_1/z_2$? *Algebraically* :



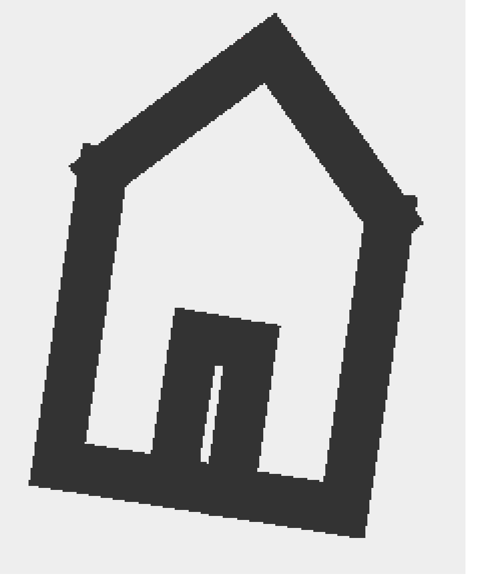
$$z = \frac{z_1}{z_2} = \frac{2 + \text{j}3}{1 + \text{j}} = \frac{(2 + \text{j}3)}{(1 + \text{j})} \cdot \frac{(1 - \text{j})}{(1 - \text{j})} =$$

$$= \frac{2 + \text{j} - 3\,\text{j}^2}{1 + 1} = \frac{5 + \text{j}}{2} = 2{,}5 + 0{,}5\,\text{j}$$

Are complex numbers used?

Complex numbers can be used for CAD and imaging and gaming applications.

A picture drawn of points in the complex plane looks the same as one that was designed in our usual two-dimensional coordinate system.



Mathematics will be easier. No complicated math functions are needed. For “twisting" and "rescaling" the image, one only has to multiply all points with a appropriate "complex" number!

The Mandelbrot set

See here a remarkable image composed of complex numbers ...

Mandelbrot set

iterate

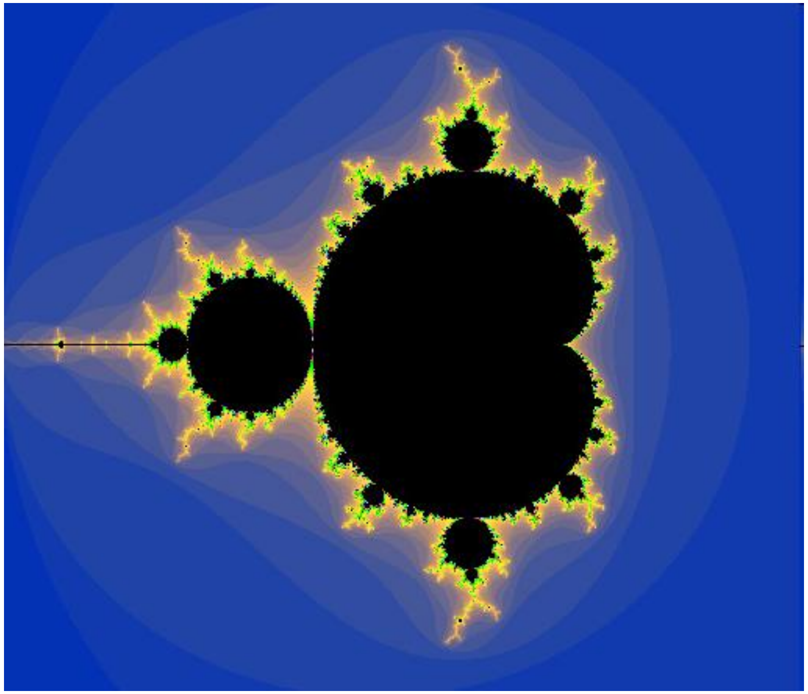
for every point a,b

$$z_{n+1} = z_n^2 + c$$

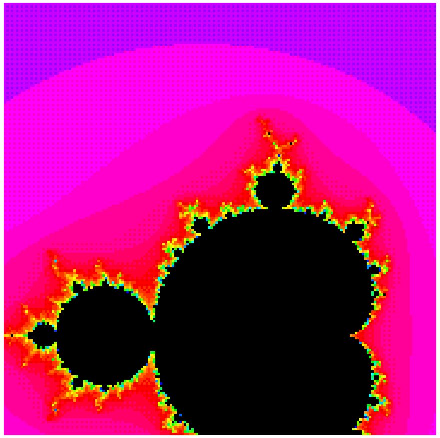
as long as

$$|z_{n+1}| < 2$$

The number of turns becomes the point color $n = color$



Mandelbrot_color_zoom.gif



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