Questions to prepare for the written exam, SF2522

(Final version)

1. Define what is meant by a probability space.
2. Define what is meant by a random variable.
3. Define what is meant by a stochastic process. Give an example of a stochastic process.
4. Formulate the basic properties of a Wiener process.
5. Define the Ito integral by the limit of the forward Euler method and show that the limit is independent of the partition used in the time discretization.
6. State a theorem on strong convergence of the forward Euler discretization applied to an Ito stochastic integral.
7. Prove that if $f : [0,T] \times \Omega \to \mathbb{R}$ is adapted and satisfies $E[|f(t+\Delta t) - f(t)|^2] \leq C\Delta t$, for some constant C and all Δt, then $E\big[\int_0^T f(s)dW(s)\big] = 0$.
8. State and prove Grönwall's lemma.
9. Show that the Ito and Stratonovich integrals (with the same integrand) may be different.
10. Show by Ito's formula that if u solves Kolmogorov's backward equation with data $u(T,x) = g(x)$, then
$$u(t,x) = E[g(X(T))|X(t) = x],$$
where X is the solution of a certain SDE (which?).
11. Formulate and prove a theorem on error estimates for weak convergence of the forward Euler method for SDE's.
12. Show that weak convergence of a sequence of random variables does not imply strong convergence.
13. State the Geometric Brownian motion equation and derive its solution.
14. State the Ornstein-Uhlenbeck equation. Derive its stationary distribution.

15. Motivate the use of Monte-Carlo methods to compute European options based on a basket of several stocks and discuss some possibilities of methods of variance reduction.

16. State and derive Ito's formula.

17. State and derive the Feynman-Kac formula.

18. State and derive the Fokker-Planck equation.

19. State and derive the central limit theorem.

20. Show how to apply the Monte Carlo method to compute an integral and discuss the error.

21. Formulate a numerical method to compute the price of an American option.

22. Derive a stability condition for a numerical method (of your choice) applied to the heat equation.

23. Present shortly an inverse problem and a method to solve it.

24. Derive the Pontryagin principle, either in its most general form, or the less general Hamiltonian system statement.

25. Formulate a numerical method for the SDE

$$\begin{aligned} dX(t) &= a(t, X(t))dt + b(t, X(t))dW(t), \\ X(0) &= x_0, \end{aligned}$$

and discuss its accuracy in case of weak and strong approximation.

26. Derive the Black-Scholes equation.

27. Discuss the alternatives to determine the option price in problem 26 in case of contracts based on one and many stocks.

28. Consider the time discrete Markov chain $X(t)$ with discrete state space and Markov control variable α. Formulate and derive a recursive relation to determine

$$\min_{\alpha \in \mathcal{A}} E_\alpha[g(X(T))].$$

29. Motivate mathematically the Hamilton-Jacobi equation for solving an optimal control problem of an ordinary differential equation.

30. Formulate and solve explicitly a portfolio optimization problem for a special utility function.

31. Formulate a metod to find

$$\inf_{\alpha \in \mathcal{A}} E[\int_0^T h(X(t), \alpha(t))dt + g(X(T))],$$

where $\alpha : [0, T] \times \mathbb{R}^n \to A$ is a control into A, which is a given compact subset of $\mathbb{R}^m$, and $\mathcal{A}$ is the set of admissible Markov control functions $t \mapsto \alpha(t, X(t))$. The state function X solves the SDE

$$dX_i = a_i(X(t), \alpha(t, X(t)))dt + b_{ij}(X(t), \alpha(t, X(t)))dW_j, \quad 0 < t < T,$$
$$X(0) = x_0.$$