



KTH Teknikvetenskap

SF1625 Envariabelanalys
Lösningsförslag till tentamen 2016-01-11

DEL A

1. We study the function f given by $f(x) = x - 2 \arctan x$.

- A. Determine the domain of definition of f .
- B. Find the intervals where f is increasing and decreasing, respectively.
- C. Find all local extreme values of f .
- D. Find all asymptotes to the graph $y = f(x)$.
- E. Sketch using the above the graph $y = f(x)$.

Lösning. A. We see that $f(x)$ is defined for all real x , and so the domain of definition is $\mathbb{R}$.

B, C. We get $f'(x) = 1 - \frac{2}{1+x^2} = \frac{x^2-1}{1+x^2}$ that exists for all x and is 0 iff $x = \pm 1$.
Hence the critical points are $x = \pm 1$.

C. We study the sign of the derivative:

If $x < -1$ then $f'(x)$ is positive. It follows that f is increasing.

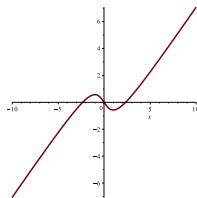
If $-1 < x < 1$ then $f'(x)$ is negative. It follows that f is decreasing.

If $x > 1$ then $f'(x)$ is positive. It follows that f is increasing.

We conclude that f has a local max at $x = -1$ and a local minimum at $x = 1$ and no other local extreme values. The function values at these points are $f(-1) = -1 + \pi/2$ and $f(1) = 1 - \pi/2$.

D. Since $\lim_{x \rightarrow \pm\infty} \arctan x = \pm\pi/2$ we see that f has an asymptote $y = x - \pi$ at infinity and an asymptote $y = x + \pi$ at minus infinity.

E. We sketch the curve:



□

Svar: A. All x . B. Increasing on $x \leq -1$. Decreasing on $-1 \leq x \leq 1$. Increasing on $x \geq 1$. C. Local max at $x = -1$ and local min at $x = 1$. D. $y = x - \pi$ at infinity and $y = x + \pi$ at minus infinity. E. Se above.

2. Compute the integrals:

A. $\int_0^{\ln 3} \frac{e^x}{1+e^x} dx$ (you may want to use the substitution $u = 1 + e^x$)

B. $\int_1^2 \frac{dx}{x^2 - 3x - 4}$ (you may want to use partial fractions)

Lösning. A. Using $u = 1 + e^x$, with $e^x dx = du$ and new interval of integration between 2 and 4, we get

$$\int_0^{\ln 3} \frac{e^x}{1+e^x} dx = \int_2^4 \frac{1}{u} du = [\ln u]_2^4 = \ln 2.$$

B. Since $x^2 - 3x - 4 = (x+1)(x-4)$ we get the partial fraction expansion

$$\frac{1}{x^2 - 3x - 4} = \frac{A}{x-4} + \frac{B}{x+1}$$

where $A = 1/5$ and $B = -1/5$. We compute the integral:

$$\int_1^2 \frac{dx}{x^2 - 3x - 4} = \int_1^2 \left(\frac{1/5}{x-4} - \frac{1/5}{x+1} \right) dx = \frac{1}{5} [\ln |x-4| - \ln |x+1|]_1^2 = \frac{2}{5} \ln \frac{2}{3}.$$

□

Svar: A. $\ln 2$

B. $\frac{2}{5} \ln \frac{2}{3}$

3. Find the solution to the differential equation

$$2y''(t) - 20y'(t) + 50y(t) = t$$

that also satisfies the initial conditions $y(0) = 1/125$ och $y'(0) = 1$.

Lösning. The solution y has the structure $y = y_h + y_p$ where y_h is the complete solution to the corresponding homogeneous equation and y_p is some particular solution.

First we find y_h . The characteristic equation $2r^2 - 20r + 50 = 0$ has the solution $r = 5$, and so we get

$$y_h(t) = (A + Bt)e^{5t}.$$

Since the right hand side is a first degree polynomial we guess $y_p(t) = ct + d$. The derivative of this is c and the second derivative is zero and when we plug that into the differential equation we obtain $-20c + 50(ct + d) = t$, hence $c = 1/50$ and $d = 1/125$.

The solution to the differential equation is therefore

$$y(t) = (A + Bt)e^{5t} + \frac{t}{50} + \frac{1}{125}.$$

The condition $y(0) = 1/125$ yields $A = 0$. The condition $y'(0) = 1$ then yields $B = 1/50$.

The solution to differential equation satisfying the conditions is therefore

$$y(t) = \frac{t}{50}e^{5t} + \frac{t}{50} + \frac{1}{125}.$$

□

Svar: $y(t) = \frac{t}{50}e^{5t} + \frac{t}{50} + \frac{1}{125}$.

DEL B

4. Compute the integral $\int_0^{1/2} \frac{1}{2+8x^2} dx$.

(For a maximum score the integral should be computed exactly, but an approximation may be given a partial score. Simplify answer.)

Lösning. We observe that:

$$\frac{1}{2+8x^2} = \frac{1}{2} \cdot \frac{1}{1+(2x)^2}.$$

Now we can compute the integral:

$$\int_0^{1/2} \frac{1}{2+8x^2} dx = \frac{1}{2} \int_0^{1/2} \frac{1}{1+(2x)^2} dx = \frac{1}{4} [\arctan(2x)]_0^{1/2} = \frac{\pi}{16}.$$

□

Svar: $\pi/16$

5. We study the equation $e^x + \arcsin x = 0$.
- A. For which real numbers x is the expression $e^x + \arcsin x$ defined?
 - B. Show that the equation $e^x + \arcsin x = 0$ has *exactly one* solution.
 - C. Find an approximate value of the solution with an error less than 0.5.

Lösning. The expression is defined for $-1 \leq x \leq 1$. Put $f(x) = e^x + \arcsin x$. The equation can then be written $f(x) = 0$. The domain of definition of f is $-1 \leq x \leq 1$ and f is continuous on the closed and bounded interval. Since $f(-1) = e^{-1} - \pi/2 < 0$ and $f(1) = e^1 + \pi/2 > 0$ the intermediate value theorem guarantees a point x^* between -1 and 1 such that $f(x^*) = 0$. In fact x^* must lie between -1 and 0 since $f(0) = 1 > 0$. An approximation of x^* with an error less than 0.5 is therefore -0.5 .

Since $f'(x) = e^x + \frac{1}{\sqrt{1-x^2}}$ is positive on the open interval, f is increasing. Therefore there cannot be more than one solution to the equation.

□

Svar: See the solution. Approximate value -0.5

6. A part of a sphere cut off by a plane is called a *spherical cap*.
- A. Compute the volume of the spherical cap obtained by letting the area between the curve $y = \sqrt{100 - x^2}$ and the x -axis, on the interval $10 - h \leq x \leq 10$, rotate around the x -axis (we assume $0 < h < 10$).
- B. A sphere with radius 10 meters is filled with water at the rate 0.2 m^3 per minute. At what speed is the surface rising at the time when the water depth h (at the deepest spot) is 2 meters? (*Hint: from the solution of problem A you get that the volume V and the depth h satisfy $V = 10h^2\pi - h^3\pi/3$.*)

Lösning. A. According to the formula for volumes of revolution the volume V is given by

$$V = \pi \int_{10-h}^{10} (100 - x^2) dx = 10h^2\pi - \frac{h^3\pi}{3}.$$

B. We use the answer to A and get that the volume V when the depth is h is given by

$$V = 10h^2\pi - \frac{h^3\pi}{3}$$

where both V and h depend on t . We differentiate with respect to t and get

$$\frac{dV}{dt} = 20h\pi \frac{dh}{dt} - h^2\pi \frac{dh}{dt}.$$

We know the the volume changes by 0.2 m^3 per minute, and so $dV/dt = 0.2$. We need to find dh/dt when $h = 2$. When $h = 2$ we see that

$$0.2 = 40\pi \frac{dh}{dt} - 4\pi \frac{dh}{dt},$$

and so

$$\frac{dh}{dt} = \frac{1}{180\pi} \text{ meters per minute.}$$

□

Svar: A. $10h^2\pi - \frac{h^3\pi}{3}$

B. $1/180\pi$ meters per minute

DEL C

7. This problem is about the theory of the concepts continuity and differentiability.
- A. Define what it means for a function to be continuous at a point a .
 - B. Define what it means for a function to be differentiable at a point a .
 - C. Prove that a function differentiable at a must be continuous at a .
 - D. Give an example showing that a continuous function does not have to be differentiable.

Lösning. See the text book.



Svar: See the text book.

8. We study the function f given by $f(x) = x^2 + \int_0^x \sin^2 t \, dt$.
- A. Compute the Taylor polynomial of degree 2 of f around the point $x = 0$.
 - B. State the error term and show that it is bounded for $|x| \leq 1$.
 - C. Compute the limit $\lim_{x \rightarrow 0} \frac{f(x)}{x^2}$.

Lösning. Using the fundamental theorem and the chain rule we get

$$f'(x) = 2x + \sin^2 x \text{ and } f'(0) = 0,$$

$$f''(x) = 2 + 2 \sin x \cos x \text{ and } f''(0) = 2,$$

$$f'''(x) = 2 \cos 2x.$$

It is clear that $f(0) = 0$ and so the Taylor polynomial is $p(x) = x^2$.

The error term is $\frac{2 \cos 2c}{3!} x^3$ for some c between 0 and x . Since $|\cos 2c| \leq 1$ the absolute value of the error is $\leq 2/3! = 1/3$ when $|x| \leq 1$. The error term is therefore bounded.

Using the above we get

$$\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = \lim_{x \rightarrow 0} \frac{x^2 + \frac{2 \cos 2c}{3!} x^3}{x^2} = 1.$$

□

Svar: A. $p(x) = x^2$. B. See the solution. C. 1

9. Find real numbers a and b such that $a \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq b$. For a maximum score your numbers should satisfy $b - a \leq 0.2$ (and you need to provide a correct argument for this).

Lösning. Clearly $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \sum_{n=3}^{\infty} \frac{1}{n^2}$.

Using estimates like those in the proof of the integral criterion it is easy to see that

$$\int_3^{\infty} \frac{1}{x^2} dx < \sum_{n=3}^{\infty} \frac{1}{n^2} < \frac{1}{9} + \int_3^{\infty} \frac{1}{x^2} dx.$$

Putting this together and computing the integral (its value is $1/3$), we get

$$1 + \frac{1}{4} + \frac{1}{3} < \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \frac{1}{4} + \frac{4}{9}.$$

Since $1 + 1/4 + 1/3 = 19/12 = 1.58\dots$ and $1 + 1/4 + 4/9 = 61/36 = 1.69\dots$ we can choose $a = 1.58$ and $b = 1.7$. Hence

$$1.58 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1.7.$$

□

Svar: Se lösningen.
