KTH Teknikvetenskap

SF1625 Envariabelanalys
Lösningsförslag till tentamen 2016-01-11

DEL A

1. Differentiate these functions with respect to x and state in each case for what x the derivative exists. Only answers are necessary, no motivations needed.

   A. $f(x) = \arctan \dfrac{1}{x}$
   B. $g(x) = 2^x$
   C. $h(x) = xe^{-x^2}$
   D. $k(x) = \dfrac{\sqrt{x}}{\ln x}$

Lösning. A. $f'(x) = \dfrac{1}{1+\frac{1}{x^2}} \cdot \left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x^2+1}$. Existerar för alla $x \neq 0$.
B. $g'(x) = 2^x \cdot \ln 2$. Existerar för alla x
C. $h'(x) = (1-2x^2)e^{-x^2}$. Existerar för alla x
D. $k'(x) = \dfrac{\frac{\ln x}{2\sqrt{x}} - \frac{1}{\sqrt{x}}}{(\ln x)^2} = \dfrac{\ln x - 2}{2\sqrt{x}(\ln x)^2}$. Existerar för alla $x > 0$ sådana att $x \neq 1$ □

Svar: Se lösningen.

2. Compute the integrals and simplify your answers.

   A. $\int \tan x\,dx$ (you may want to use the substitution $u = \cos x$)

   B. $\int x^2 \cos x\,dx$ (you may want to use repeated integration by parts)

Lösning. A. We use $u = \cos x$ med $du = -\sin x\,dx$ and obtain

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x} x\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C$$

C an arbitrary constant.

B. We integrate by parts twice and obtain

$$\begin{aligned}\int x^2 \cos x\,dx &= x^2 \sin x - \int 2x \sin x \\ &= x^2 \sin x + 2x\cos x - \int 2\cos x \\ &= x^2 \sin x + 2x \cos x - 2\sin x + C.\end{aligned}$$

□

Svar: A. $-\ln|\cos x| + C$, C an arbitrary constant.

B. $x^2 \sin x + 2x\cos x - 2\sin x + C$, C an arbitrary constant.

3. Decide whether the function $f(x) = |2x-1| + \arcsin x$ assumes maximum and minimum values, and if so, find these. Simplify your answer.

Lösning. Domain $-1 \le x \le 1$. We oberve that f is continuous on the closed and bounded interval and so a maximum and a minimum value exist, and are assumed at either a critical point or a boundary point or a singular point.

$$f(x) = \begin{cases} 2x - 1 + \arcsin x, & \text{if } 1/2 \le x \le 1 \\ 1 - 2x + \arcsin x & \text{if } -1 \le x < 1/2 \end{cases}$$

We differentiate:

$$f'(x) = \begin{cases} 2 + \frac{1}{\sqrt{1-x^2}}, & \text{om } 1/2 < x < 1 \\ -2 + \frac{1}{\sqrt{1-x^2}} & \text{om } -1 < x < 1/2 \end{cases}$$

At $x = 1/2$ the function is not differentiable. We see that $f'(x) > 0$ on $1/2 < x < 1$ and so f has no critical points in that interval. On $-1 < x < 1/2$ it holds that $f'(x) = 0$ iff $x = -\sqrt{3}/2$, which therefore is the only critical point.

It follows from the above that the maximum and minimum values of f is assumed at -1, $-\sqrt{3}/2$, $1/2$ or 1. We compare values:

$$f(-1) = 3 - \frac{\pi}{2}, \quad f(-\sqrt{3}/2) = \sqrt{3} + 1 - \frac{\pi}{3}, \quad f(1/2) = \frac{\pi}{6}, \quad f(1) = 1 + \frac{\pi}{2}.$$

We see f:s maximum value is $1 + \pi/2$ and f:s minimum value is $\pi/6$ □

Svar: Maximum value $1 + \pi/2$ and minimium value $\pi/6$

DEL B

4. Assume the function f to be three times differentiable on the real axis. Assume further that $f(1) = 2$, $f'(1) = -3$ and $|f''(x)| \leq 5$ for all x.
   A. Find an approximate value of $f(1.1)$ using linear approximation (Taylor polynomial of degree 1).
   B. Find as good a bound as possible for the error in your approximation.

Lösning. Using Taylors formula we get

$$f(x) \approx 2 - 3(x-1), \qquad \text{for } x \text{ near } 1.$$

The error is $\dfrac{f''(c)}{2!}x^2$ for some c between 1 and x.

With $x = 1.1$ this yields

$$f(1.1) \approx 2 - 3(1.1 - 1) = 1.7.$$

The sought for approximate value is 1.7.

The error is at most $\dfrac{5}{2!}0.1^2 = 0.025$. □

Svar: A. 1.7. B. 0.025

5. Compute the integral

$$\int_0^{\sqrt{3}} \arctan x \, dx.$$

(For a maximum score the integral should be computed exactly, but an approximate computation may be awarded partial score. Simplify your answer.)

Lösning. We integrate by parts and obtain

$$\begin{aligned}\int_0^{\sqrt{3}} \arctan x \, dx &= [x \arctan x]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2} \, dx \\ &= \frac{\sqrt{3}\pi}{3} - [(1/2)\ln(1+x^2)]_0^{\sqrt{3}} \\ &= \frac{\sqrt{3}\pi}{3} - \ln 2.\end{aligned}$$

(If you want to approximate the integral instead you may use for instance Taylors formula, Riemann sums or the trapezoid rule)

□

Svar:
$\frac{\sqrt{3}\pi}{3} - \ln 2$

6. We study the curve given by $2x^2 + 4xy + 3y^2 + 2y = 10$.
   A. Find an equation for the tangent to the curve at the point $(x_0, y_0) = (-1, 2)$.
   B. Using the tangent, find an approximate value of the y-coordinate of a point on the curve with x-coordinate -0.8.
   C. Can there be more than one point on the curve with x-coordinate -0.8?

Lösning. A. The poin $(-1, 2)$ satisfies the equation, hence it lies on the curve. We assyme $y = y(x)$ and differentiate implicitly and obtain

$$4x + 4y(x) + 4xy'(x) + 6y(x)y'(x) + 2y'(x) = 0$$

which if $x = -1$ and $y = 2$ yields

$$-4 + 8 - 4y'(-1) + 12y'(-1) + 2y'(-1) = 0.$$

We solve for $y'(-1)$ and get

$$4 + 10y'(-1) = 0$$

in other words $y'(-1) = -2/5$. The equation of the tangent is

$$y - 2 = -\frac{2}{5}(x + 1)$$

B. We use the tangent for approximation and get

$$y \approx 2 - \frac{2}{5}(-0.8 + 1) = 2 - \frac{0.4}{5} = 1.92.$$

C. With $x = -1$ we use the solution formula for the quadratic equation and get

$$2 - 4y + 3y^2 + 2y = 10 \iff y = \frac{1}{3} \pm \frac{5}{3}$$

which means there are two points on the curve with x-coordinat -1. (Ellipse) □

Svar: A. $y - 2 = -\frac{2}{5}(x + 1)$. B 1.92. C. yes

DEL C

7. We study the function f given by

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}$$

A. Is f odd? Is f even?
B. At what points is f continuous?
C. At what points is f differentiable?
D. Is f integrable on the interval $-\pi \leq x \leq \pi$?

Lösning. A. Since $f(0) \neq 0$, f cannot be odd. We see that for $x \neq 0$ it holds that $f(-x) = \sin(-x)/(-x) = \sin x/x = f(x)$ and so f is even.

B. For all $x \neq 0$, f is given by an elementary expression and so is continuous. Since

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1 = f(0)$$

f is continuous at $x = 0$ too. Hence f is continuous everywhere.

C. For $x \neq 0$ we get

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

and so f is differentiable at all $x \neq 0$. The point 0 has to be examined separately using the definition of the derivative:

$$f'(0) = \lim_{h\to 0} \frac{\frac{\sin h}{h} - 1}{h} = \lim_{h\to 0} \frac{h^2/3! + \mathcal{O}(h^4)}{h} = 0.$$

We see f is differentiable at the origin and $f'(0) = 0$. Hence f is differentiable everywhere.

D. Since f is continuous on the closed and bounded interval $[-\pi, \pi]$ it follows that f is integrable on that interval. □

Svar: See solution

8. Find the smallest possible real number M such that $|f''(x)| \leq M$ for all $x \in \mathbb{R}$, if $f(x) = \arctan x$.

Lösning. With $f(x) = \arctan x$ we get $f'(x) = \dfrac{1}{1+x^2}$ and $f''(x) = \dfrac{-2x}{(1+x^2)^2}$ that exists for all real x.

We shall find the maximum value of the function g given by $g(x) = \dfrac{-2x}{(1+x^2)^2}$ for $x \in \mathbf{R}$. We differentiate:

$$g'(x) = -2\frac{(1+x^2)^2 - (1+x^2)4x^2}{(1+x^2)^4},$$

that exists for all real x. We see that $g'(x) = 0 \iff x = \pm 1/\sqrt{3}$. We study the sign of g' and get:

If $x < -1/\sqrt{3}$ then $g'(x)$ is positive. Conclusion: g is increasing here.

If $-1/\sqrt{3} < x < 1/\sqrt{3}$ then $g'(x)$ is negative. Conclusion: g is decreasing here.

If $x > 1/\sqrt{3}$ then $g'(x)$ is positive. Conclusion: g is increasing here.

Since $\lim_{x\to\pm\infty} g(x) = 0$ it follows that g assumes its maximum value at $-1/\sqrt{3}$, where $g(-1/\sqrt{3}) = 2\sqrt{3}/(4/3)^2 = 3\sqrt{3}/8$, and its minimum value at $1/\sqrt{3}$, where $g(1/\sqrt{3}) = -3\sqrt{3}/8$.

Therefore $M = 3\sqrt{3}/8$ is the smallest number such that $|f''(x)| \leq M$ for all $x \in \mathbf{R}$ □

Svar: $3\sqrt{3}/8$

9. The curves $y = x^{2/3}$ and $y = x^{3/2}$ bound a domain in the first quadrant. Compute the length of the boundary curve of that domaih.

   Lösning. The boundary curve consists of two parts intersecting when $x^{2/3} = x^{3/2}$ i.e. when $x = 0$ and when $x = 1$. Since $f(x) = x^{2/3}$ and $g(x) = x^{3/2}$ are inverse to each other (check that $f(g(x)) = g(f(x)) = x$) the two parts have equal length. It is therefore enough to compute the length of one of them, for instance $y = x^{3/2}$ when $0 \le x \le 1$. The length is:

$$L = \int_0^1 \sqrt{1 + (y'(x))^2}\, dx = \int_0^1 \sqrt{1 + \frac{9}{4}x}\, dx = \left[\frac{(1 + \frac{9}{4}x)^{3/2}}{27/8} \right]_0^1 = \frac{13^{3/2} - 8}{27}$$

   The length of the boundary curve is $2L$, i.e.

$$2\frac{13^{3/2} - 8}{27}.$$

□

Svar: $2\frac{13^{3/2}-8}{27}$