## SF2729 Groups and Rings

Exam
Wednesday, June 8, 2016

Time: 08:00-13:00
Allowed aids: none
Examiner: Roy Skjelnes
Present your solutions to the problems in a way such that arguments and calculations are easy to follow. Provide detailed arguments to your answers. An answer without explanation will be given no points.

The final exam consists of six problems, each of which can give up to 6 points. The homework problems will contribute with up to 9 points on the three first problems in the exam. Credits on the three first problems in the exam will together with the contribution from the homeworks give at most 18 points.

The minimum scores required for each grade are given by the following table:

| Grade | A | B | C | D | E | Fx |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: |
| Credit | 30 | 27 | 24 | 21 | 18 | 16 |

A score of 16 or 17 is a failing grade with the possibility to improve to an E grade by additional work.

## Problem 1

1. Define the concept Sylow p-subgroup.
2. Formulate Sylow's theorem.
3. Use Sylow's theorem to prove that there is a unique group of order 33 .

## Problem 2

Let $f: R \rightarrow S$ be a homomorphism of commutative unitary rings where $f(1)=1$.

1. Give the definition of an ideal being a prime ideal.
2. Let $P$ be a prime ideal in $S$, show that $f^{-1}(P)$ is prime.
3. Let $P$ be a maximal ideal in $S$. Is $f^{-1}(P)$ necessarily maximal?

## Problem 3

Let $G$ be a finite group acting on a finite set $X$. Burnside's theorem states that the number of orbits is $\frac{1}{|G|} \sum_{g \in G}\left|X_{g}\right|$, where $X_{g}=\{x \in X \mid g \cdot x=x\}$.

1. Show that $\sum_{g \in G}\left|X_{g}\right|=\sum_{x \in X}\left|G_{x}\right|$. (Hint: consider $\{(g, x) \mid g \cdot x=x\} \subseteq G \times X$ )
2. Prove Burnside's Theorem.
3. The symmetric group on three letters acts naturally on the set $X$ of 64 (equilateral) triangles having each edge painted with one of four colours. Apply Burnside's theorem to determine the number of distinguishable triangles.

## Problem 4

Let $R=\mathbb{Z}[x] /\left(x^{2}+1\right)$ be the ring of Gaussian integers.

1. Determine the maximal ideal $I$ such that $I^{2}=(2)$.
2. Show that the quotient ring $R /(2+3 x)$ is a field with 13 elements.

## Problem 5

Let 1 denote the trivial group. A short exact sequence of groups is a sequence of groups and homomorphisms

$$
1 \xrightarrow{\varphi_{1}} G \xrightarrow{\varphi_{2}} H \xrightarrow{\varphi_{3}} K \xrightarrow{\varphi_{4}} 1,
$$

where the kernel of a homomorphism equals the image of the homomorphism preceeding it, that is $\operatorname{ker}\left(\varphi_{i+1}\right)=\operatorname{im}\left(\varphi_{i}\right)$ for any $i=1,2,3$.

1. Show that if $1 \rightarrow G \rightarrow H \rightarrow K \rightarrow 1$ is a short exact sequence and $H$ is finite, then both $G$ and $K$ are finite and $|H|=|G| \cdot|K|$.
2. Determine whether for odd integers $n \geq 3$ there is a short exact sequence

$$
1 \rightarrow C_{2} \rightarrow D_{2 n} \rightarrow C_{n} \rightarrow 1
$$

where $D_{2 n}$ is the dihedral group, and $C_{n}$ is the cyclic group, and their order is given by their respective indices.
3. Determine whether for odd integers $n \geq 3$ there is a short exact sequence

$$
1 \rightarrow C_{n} \rightarrow D_{2 n} \rightarrow C_{2} \rightarrow 1
$$

## Problem 6

1. In $R=\mathbb{Z}[x] /\left(x^{2}-7\right)$ we have the equality $2 \cdot 3=(x+1)(x-1)$. Does this equality imply that $R$ is not a UFD?
2. Show that $\mathbb{Z}[x] /\left(x^{2}+d\right)$ is not a UFD when $d \geq 3$.
