

# Reglerteknik: Exercises

- Exercises, Hints, Answers
- Liten reglerteknisk ordlista
- Introduktion till Control System Toolbox ver. 5

This version: August 16, 2016



# Exercises

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# 1 Mathematics

1.1 Determine the one-sided Laplace transform of the following signals.

a)  $u(t) = \begin{cases} 0, & t < 0 \\ A, & t \geq 0 \end{cases}$ , where  $A$  is a constant.

b)  $u(t) = \begin{cases} 0, & t < 0 \\ At, & t \geq 0 \end{cases}$ , where  $A$  is a constant.

c)  $u(t) = e^{-2t}$  for  $t \geq 0$ .

d)  $u(t) = \cos 5t$  for  $t \geq 0$ .

Express the following in  $U(s)$ , the Laplace transform of  $u(t)$ .

e)  $\dot{u}(t)$

f)  $\dot{u}(t)$ , when  $u(t) = 0$  for  $t \leq 0$ .

g)  $\ddot{u}(t)$

h)  $\ddot{u}(t)$ , when  $u(t) = \dot{u}(t) = 0$  for  $t \leq 0$ .

i)  $u(t - T)$

1.2 Consider the differential equation

$$\dot{y}(t) + 2y(t) = u(t)$$

a) If  $u(t)$  is constant then  $\dot{y}(t) \approx 0$  when time goes to infinity. What value will  $y(t)$  approach as  $t \rightarrow \infty$  if  $u(t) = 5$ ?

b) Determine the transfer function relating  $U(s)$  and  $Y(s)$  for the differential equation above.

1.3 Determine the general solution of the differential equation

$$\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = 3\sin(2t)$$

1.4 Below, differential equations that describe dynamic systems are given together with system inputs and initial conditions. Use the Laplace transform to determine the system outputs.

a)

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sigma(t)$$

$$\sigma(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

$$\frac{dy}{dt}(0) = y(0) = 0$$

b)

$$\dot{y}(t) + y(t) = u(t)$$

$$u(t) = 1 + \sin t$$

$$y(0) = 0$$

1.5 Write the following complex numbers in polar form, that is, determine their absolute value and argument.

a)  $1 + i$

b)  $\frac{1+i}{5i(1+\sqrt{3}i)}$

Write the following complex numbers on rectangular form:

c)  $2e^{i\frac{\pi}{3}}$

d)  $5e^{-i\pi}$

1.6 A system has amplification 100. What is the amplification expressed in decibel ( $\text{dB}_{20}$ )? What is the amplification corresponding to  $20 \text{ dB}_{20}$ ,  $-3 \text{ dB}_{20}$ ,  $0 \text{ dB}_{20}$ ,  $-10 \text{ dB}_{20}$ , and  $10 \text{ dB}_{20}$  respectively?

1.7 Verify that the following rule for inversion of  $2 \times 2$  matrices holds.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

1.8 Determine eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -6 & 2 & 1 \end{pmatrix}$$

1.9 Determine a transformation matrix  $T$ , such that  $T^{-1}AT$  is a diagonal matrix, where

$$A = \frac{1}{3} \begin{pmatrix} 6 & 0 & -3 \\ -1 & 5 & -1 \\ -2 & -2 & 7 \end{pmatrix}$$

1.10 Characterize the range space and null space of the matrix  $A$  by specifying bases for them. Find the rank of the matrix.

$$A = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 3 & 1 & 3 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

1.11 What are the time functions corresponding to the Laplace transforms below? What values will the time functions approach as time goes to infinity?

a)

$$F(s) = \frac{1}{s^2 + s}$$

b)

$$F(s) = \frac{1}{s^2 - 1}$$

c)

$$F(s) = \frac{1}{(s + 1)^2}$$

1.12 The water level,  $y$ , in a tank is modelled by the differential equation

$$\dot{y}(t) + y(t) = z(t)$$

where  $z$  denotes the inflow. The inflow is a function of a valve position, which in turn is controlled by the electric control signal  $u$ . The relation between control signal and flow is given by the differential equation

$$\ddot{z}(t) + \dot{z}(t) + z(t) = u(t)$$

What differential equation relates the water level  $y$  to the control signal  $u$ ?

## 2 Dynamic Systems

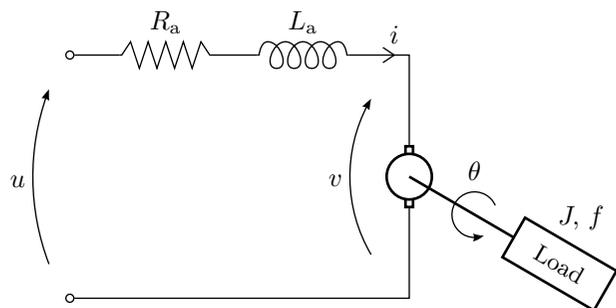


Figure 2.1a

2.1 A common component in a control system is a DC-motor. A schematic picture of the motor is shown in Figure 2.1a. The motor is characterized by a number of physical relationships as will now be explained. The rotating axis is described by

$$J\ddot{\theta} = -f\dot{\theta} + M,$$

where  $\theta$  is the angle of rotation,  $M$  is the torque,  $J$  is the moment of inertia of the load and  $f$  is the frictional coefficient.

The interplay between rotor and stator is given by

$$M = k_a i \quad \text{and} \quad v = k_v \dot{\theta}$$

where  $i$  is the anchor current,  $k_a$  a proportional constant characteristic for the motor,  $v$  is voltage induced by the rotating axis and  $k_v$  is a proportional constant. The input voltage  $u$  is the control signal and  $\theta$  is the output.

- Use the equations above and Kirchhoff's voltage law to write a differential equation that relates  $u$  and  $\theta$ . The inductance  $L_a$  can be neglected.
- Determine the transfer function of the system from  $u$  to  $\theta$ .
- Study the behavior of the system by calculating  $\theta$  when  $u$  is a step.

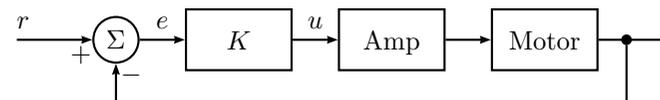


Figure 2.2a

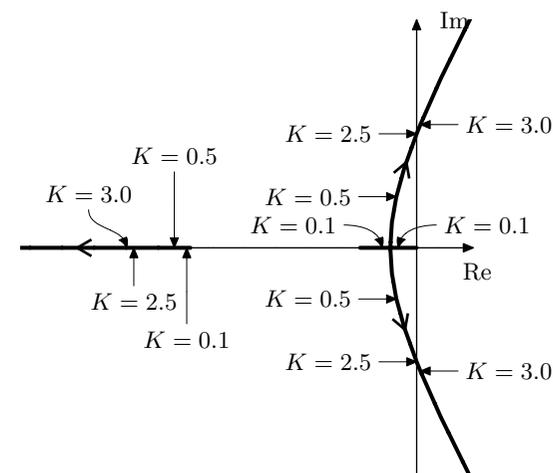


Figure 2.2b

2.2 A servo system for positioning of a tool in a tooling machine is depicted in Figure 2.2a. In Figure 2.2b, the poles of the closed loop system are plotted for different values of the gain  $K$ . Find (without calculations), for each of the step responses in Figure 2.2c, the corresponding value of  $K$  used when generating the step response.

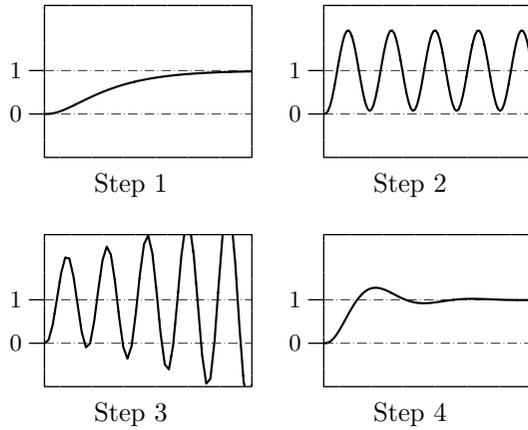


Figure 2.2c. All comparable axes have equal scaling.

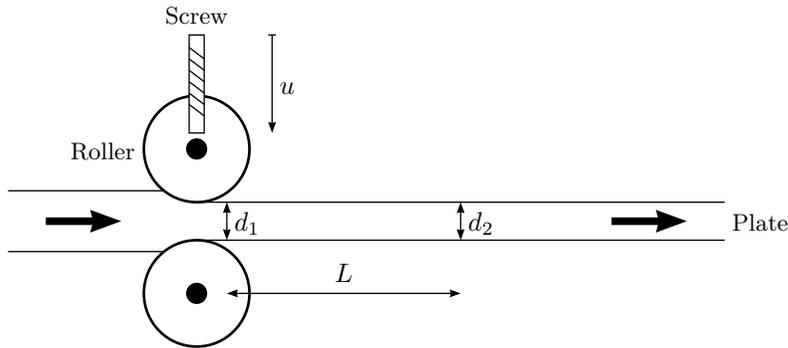


Figure 2.3a

2.3 Consider the simple model of the roller depicted in Figure 2.3a. To obtain a simple model we describe the relationship between the position of the screw and the thickness of the sheet  $d_1$  directly after the rollers as a first order transfer function

$$G(s) = \frac{\beta}{1 + sT}$$

To determine the constants  $\beta$  and  $T$  we register the effect of a sudden change in the position of the screw. The units used in the model are chosen such that a unit step will make an appropriately sized input for identification purposes,

and that is the input used in the experiment for which the resulting thickness profile  $d_1(t)$  is shown in Figure 2.3b. In production the thickness cannot be measured directly behind the rollers for practical reasons, and instead the thickness  $d_2(t)$  is measured  $L$  length units after the rollers. Find the transfer function from the position of the screws to the thickness  $d_2$ . The sheet moves with speed  $V$ .

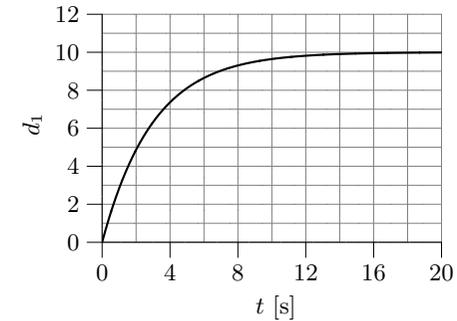


Figure 2.3b

2.4 The step response of the following system

$$G(s) = \frac{1}{s^2 + s + 1}$$

is shown as the dashed line in each part of Figure 2.4a.

a) The step response of the system

$$G(s) = \frac{1}{s^2 + as + 1}$$

is shown as the solid line in the left of Figure 2.4a. Determine if  $a > 1$  or  $a < 1$ .

b) The step response of the system

$$G(s) = \frac{b^2}{s^2 + bs + b^2}$$

is shown as the solid line in the right of Figure 2.4a. Find  $b$ .

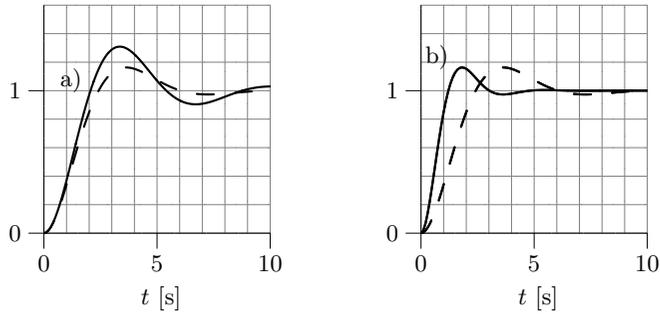


Figure 2.4a. Dashed: original system. Solid left: part a). Solid right: part b).

2.5 Pair the step responses and pole-zero diagrams in Figure 2.5a.

2.6 Consider the systems

$$G_A(s) = \frac{1}{s^2 + 2s + 1} \quad G_B(s) = \frac{1}{s^2 + 0.4s + 1}$$

$$G_C(s) = \frac{1}{s^2 + 5s + 1} \quad G_D(s) = \frac{1}{s^2 + s + 1}$$

$$G_E(s) = \frac{4}{s^2 + 2s + 4}$$

- Use MATLAB to plot the step responses of the systems. Find  $T_r$  (rise time),  $T_s$  (settling time) and  $M$  (overshoot) for the five step responses.
- Compute the poles of the systems  $G_A(s)$ ,  $G_B(s)$ ,  $G_C(s)$ ,  $G_D(s)$ , and  $G_E(s)$  respectively.
- How is the location of the poles related to the properties of the step responses?

2.7 Consider a system with the transfer function

$$G(s) = \frac{\alpha s + 1}{s^2 + 2s + 1}$$

Compute and plot the step response of the system for some different values of  $\alpha$  in the range  $-10 < \alpha < 10$ . How are the properties of the step response affected by the location of the zero of the system?

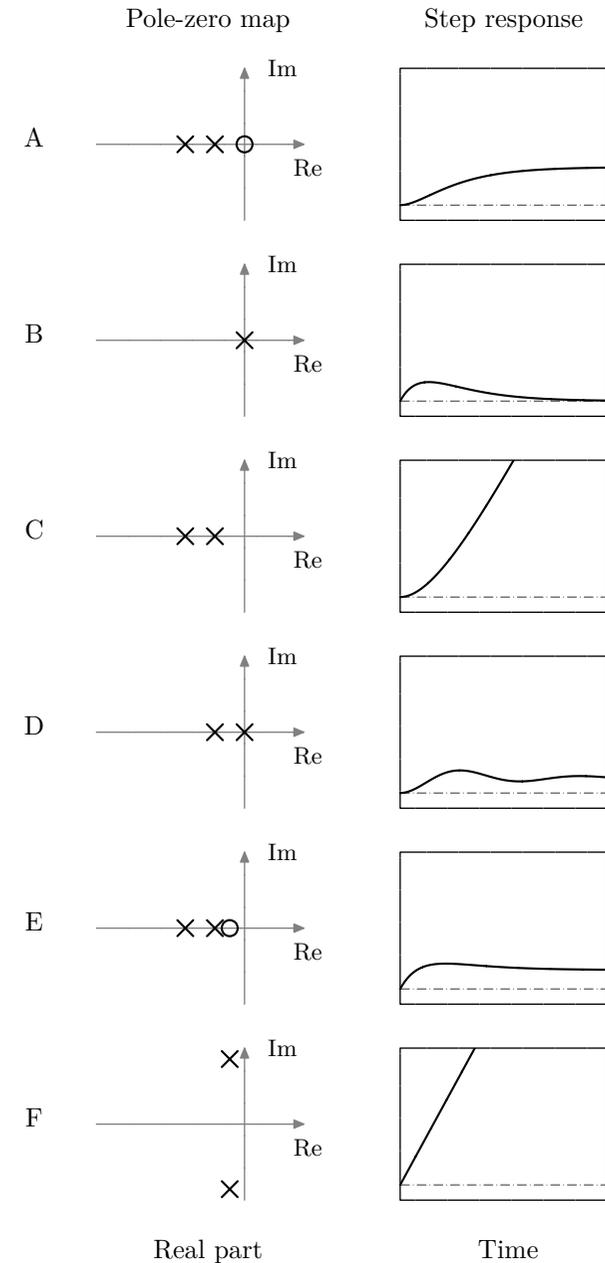


Figure 2.5a. All comparable diagrams have equal scaling. In the pole-zero maps, imaginary and real parts have equal scaling,  $\times$  marks poles, and  $\circ$  marks zeros.

2.8 Consider a system described by the model  $Y(s) = G(s)U(s)$  as shown in Figure 2.8a. Given  $G(s)$ , how is the step response computed? How can the step response be determined using experiments?

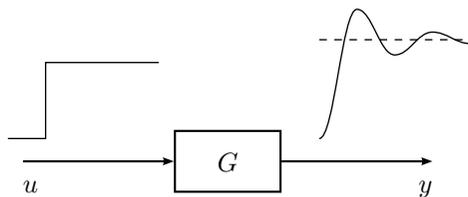


Figure 2.8a

2.9 Figure 2.9a shows the step response of a system  $Y(s) = G(s)U(s)$ . The input step has amplitude 1. Use the figure and determine

- Steady state value.
- Overshoot  $M$  in % of the final value.
- Rise time  $T_r$ .
- Settling time  $T_s$ .

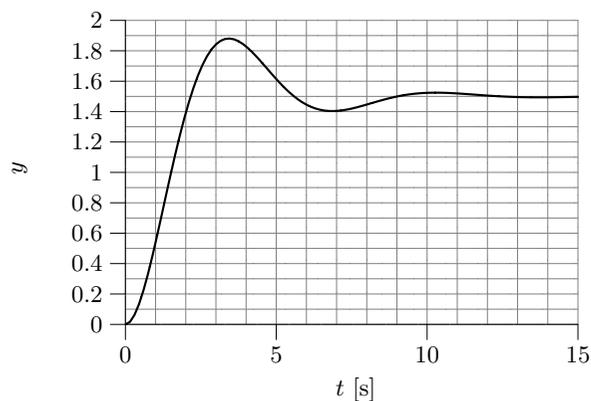


Figure 2.9a

2.10 Figure 2.10a shows the step responses of four different systems. Combine each step response with a transfer function from the alternatives below.

Transfer function	Poles	Zeros	$ G(0) $
$G_1(s) = \frac{100}{s^2+2s+100}$	$-1 \pm 10i$		1
$G_2(s) = \frac{1}{s+2}$	-2		1/2
$G_3(s) = \frac{10s^2+200s+2000}{(s+10)(s^2+10s+100)}$	$-10, -5 \pm 8.7i$	$-10 \pm 10i$	2
$G_4(s) = \frac{200}{(s^2+10s+100)(s+2)}$	$-2, -5 \pm 8.7i$		1
$G_5(s) = \frac{600}{(s^2+10s+100)(s+3)}$	$-3, -5 \pm 8.7i$		2
$G_6(s) = \frac{400}{(s^2-10s+100)(s+2)}$	$-2, 5 \pm 8.7i$		2

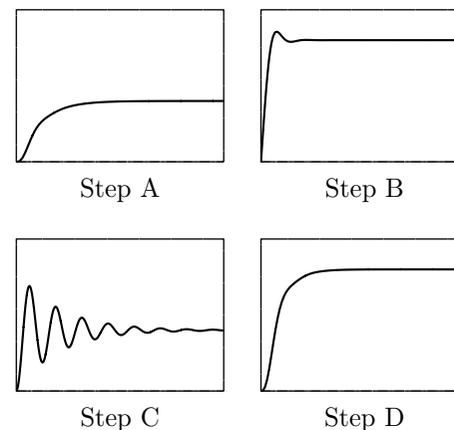


Figure 2.10a. All comparable axes have equal scaling.

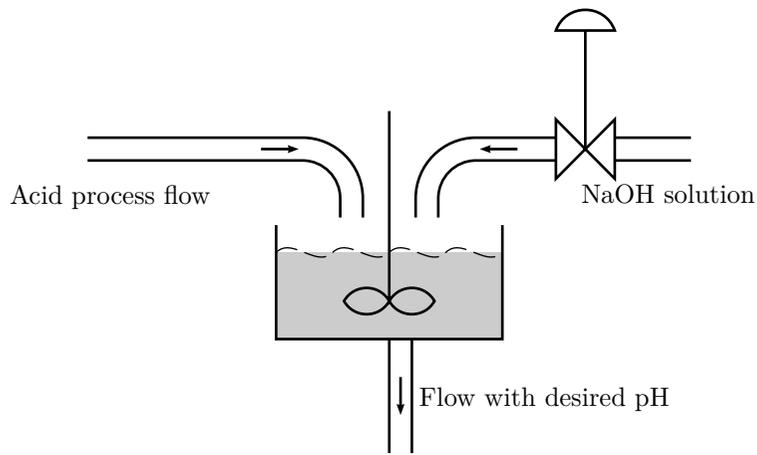


Figure 2.11a

2.11 In the continuously stirred tank, see Figure 2.11a, an acid process flow is neutralized by adding a concentrated NaOH solution. The acid process flow has a tendency to vary its pH with time, which gives undesired variation of the pH in the outflow. In an effort to reduce these variations one has decided to use control.

- Classify the different signals as input, output, and disturbance signal.
- Draw a block diagram of the system with a control strategy.

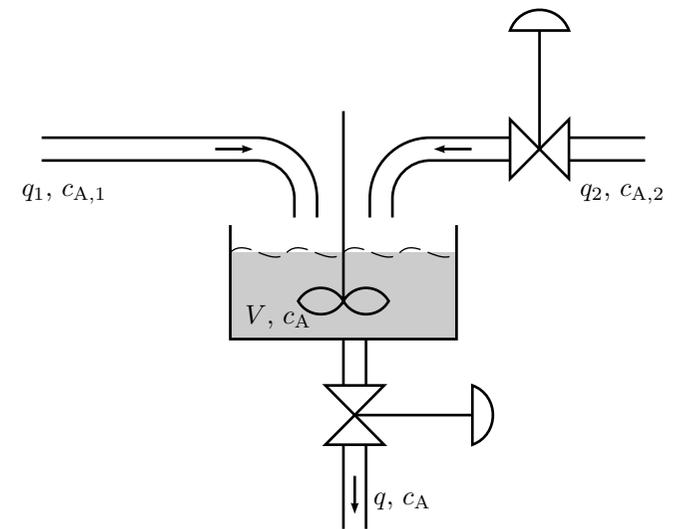


Figure 2.12a

2.12 Two flows with different concentrations of a chemical component A are mixed in a continuously stirred tank, see Figure 2.12a. The volume can be assumed constant,  $V = 1 \text{ m}^3$ .

- During a stationary period the values of the concentrations and flows are,  $c_{A,1}^* = 1.0 \text{ kmol/m}^3$ ,  $q_1^* = 1.0 \text{ m}^3/\text{min}$ ,  $c_{A,2}^* = 4.0 \text{ kmol/m}^3$  and  $q_2^* = 0.5 \text{ m}^3/\text{min}$ . What are  $c_A(t)$  and  $q(t)$  during this period?
- Write down the dynamical balance equation for component A. State all your assumptions. Is the dynamical model linear?
- Assume that  $c_{A,1}$  changes value from  $1.0 \text{ kmol/m}^3$  to  $1.2 \text{ kmol/m}^3$  at  $t = 0$ . Show that the expression for  $c_A(t)$  can be written as

$$c_A(t) = k_0 + k_1 (1 - e^{-t/\tau})$$

Determine the constants  $k_0$ ,  $k_1$  and  $\tau$ .

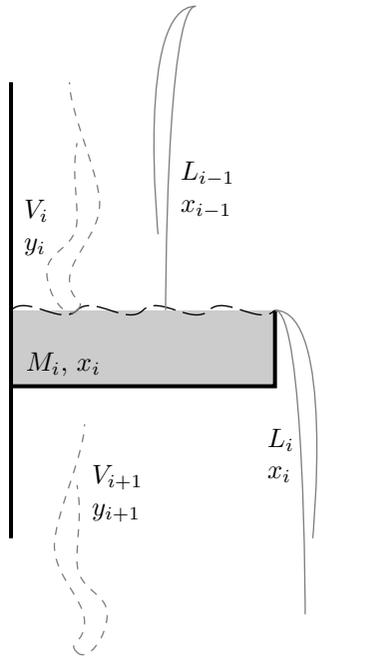


Figure 2.13a

2.13 The distillation plate, see Figure 2.13a, has the following variables:

$L_i$  - Liquid flow from plate  $i$  (kmol/min)

$V_i$  - Steam flow from plate  $i$  (kmol/min)

$M_i$  - Amount of liquid on plate  $i$  (kmol)

$x_i$  - Mole fraction of the most volatile component in the liquid on plate  $i$ .

$y_i$  - Mole fraction of the most volatile component in the steam from plate  $i$ .

Assumptions:

- Perfect mixing of liquid on the plate.
- Binary separation
- $y_i$  is in equilibrium with  $x_i$ .

- The equilibrium is described by

$$y_i = \frac{\alpha x_i}{1 + (\alpha - 1)x_i}$$

- a) Write the two differential equations that describe  $M_i(t)$  and  $x_i(t)$ .
- b) Linearize the model under the assumption that the change of the mass on the plate is zero.

2.14 Figure 2.14a shows the step response from a system with the transfer function

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

for the following sets of values. Combine values and step responses.

i)  $\omega_0 = 4 \quad \zeta = 0.5$

ii)  $\omega_0 = 4 \quad \zeta = 0.3$

iii)  $\omega_0 = 2 \quad \zeta = 0.5$

iv)  $\omega_0 = 2 \quad \zeta = 0.3$

2.15 **2009-12-15 Uppg. 1d**

Beräkna överföringsfunktionen från  $U(s)$  till  $Y(s)$  för systemet i figur 2.15a.

2.16 **2009-12-15 Uppg. 1d**

I denna deluppgift ska vi studera de sex överföringsfunktionerna

$$G_A(s) = \frac{s+1}{s^2+s+1}, \quad G_B(s) = \frac{1-s}{s^2+s+1}, \quad G_C(s) = \frac{s+1}{s^2+0.3s+1}$$

$$G_D(s) = \frac{1}{5s}, \quad G_E(s) = \frac{1}{s^2+2s+1}, \quad G_F(s) = \frac{9}{s^2+6s+9}$$

I Figur 2.16a syns de sex stegsvaren för överföringsfunktionerna ovan. Para ihop rätt stegsvar och överföringsfunktion. Skalorna på både x-axlarna (tid) och y-axlarna (utsignal) är lika för alla figurerna. Rätt svar ger +1p, fel svar ger -1p, och inget svar ger 0p.

Motivera noga! (Ej fullständig motivering räknas som felaktigt svar.)

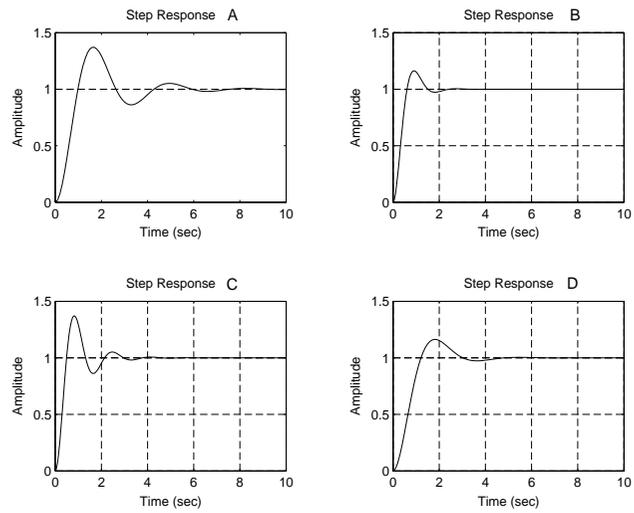


Figure 2.14a

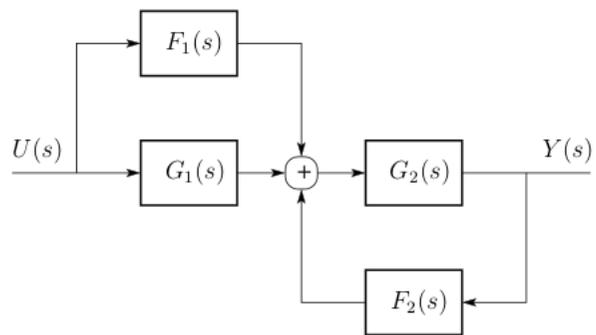


Figure 2.15a

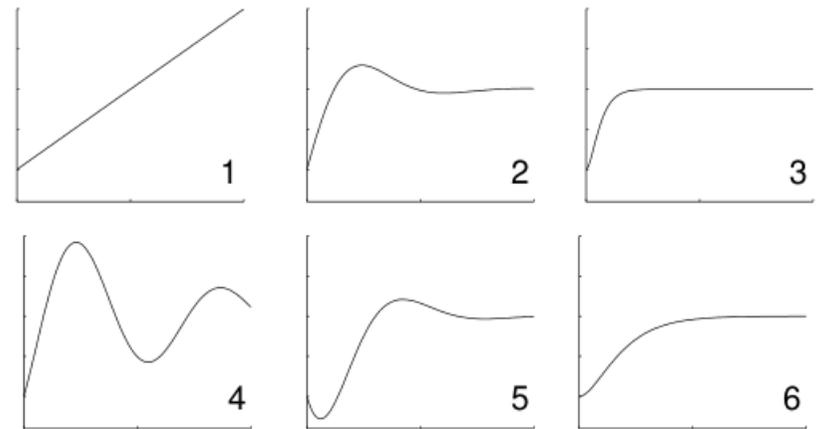


Figure 2.16a

### 3 Feedback Systems

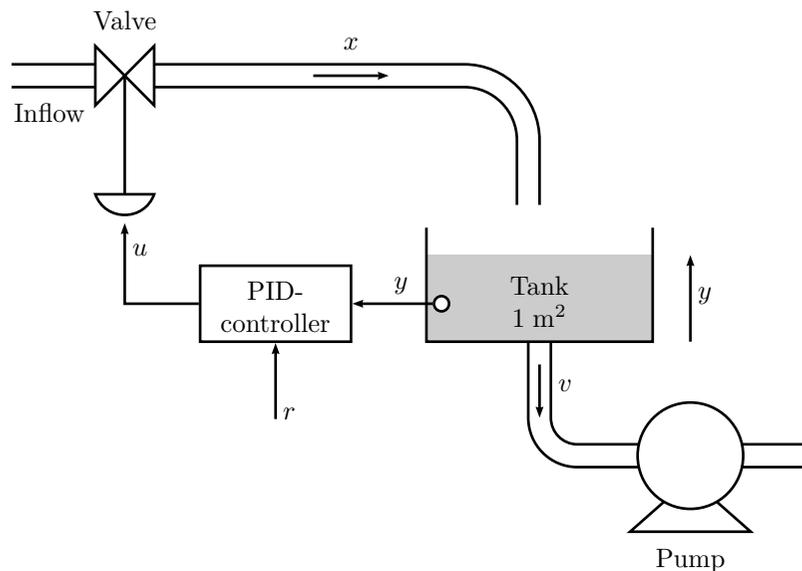


Figure 3.1a

3.1 A feedback system for level control is shown in Figure 3.1a where all variables denote variations from a working point. The flow to the tank is given by the valve position and the outflow from the tank by the flow  $v(t)$  via the pump. The transfer function from valve opening  $u$  to the flow  $x$  is denoted  $G_v(s)$ .

- a) Determine the important signals of the system and draw a block diagram of the whole system. Use mass balance\* to determine a transfer function for the tank.
- b) The transfer function of the valve is

$$G_v(s) = \frac{k_v}{1 + Ts}$$

To find  $k_v$  and  $T$  a unit step change in  $u$  has been applied. The step response is shown in Figure 3.1b. Determine the constants  $k_v$  and  $T$ .

\*That the change in tank level is proportional to the difference between inflow and outflow.

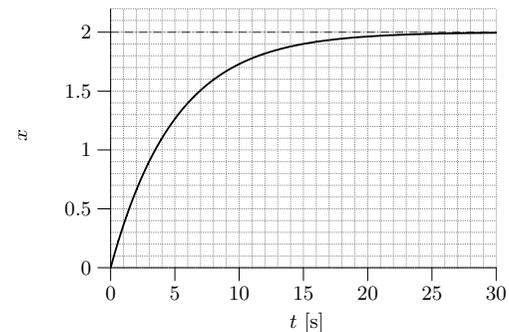


Figure 3.1b

- c) Compute the transfer functions from  $h_{ref}$  to  $h$  and from  $v$  to  $h$  and verify that they have the same poles.
- d) Assume that we use proportional control, that is,  $F(s) = K$ . How large gain,  $K$ , can we select if we want all poles of the closed loop system to be in the area shown in Figure 3.1c?
- e) Assume that a disturbance is introduced in the outflow  $v$  in the form of a unit step. How large will the level error due to the disturbance be in steady state with control according to d)?
- f) How large will the steady state level error due to the disturbance be with a PI controller?

3.2 Consider the tank system in Problem 3.1.

- a) Assume that proportional control with gain  $K = 1$  has been selected. Which poles will the closed loop system have?
- b) Consider a PD controller

$$u = -K_P h - K_D \frac{dh}{dt}$$

Assume that  $K_P = 1$  and calculate a value of  $K_D$  so the damping ratio of the closed loop poles will be greater than  $1/\sqrt{2}$ . This corresponds to the

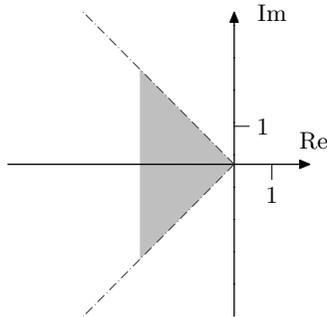


Figure 3.1c

grey area in Figure 3.1c. Poles in this region corresponds to an overshoot of less than 5% (the overshoot may also be affected by the zeros).

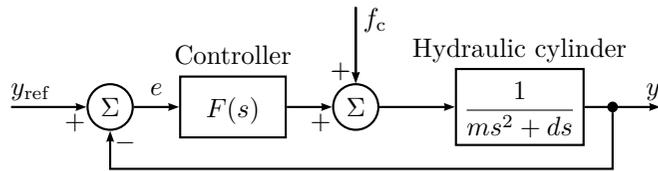


Figure 3.3a

3.3 Figure 3.3a shows the block diagram for a hydraulic servo system in an automatic lathe (Swedish: *svarv*). The output signal  $y(t)$  represents the position of the lathe tool (Swedish: *svarvstål*),  $y_{\text{ref}}(t)$  is the desired position of the lathe tool,  $m$  is the mass of the tool slide (Swedish: *verktygsslid*) and the hydraulic piston (Swedish: *kolv*),  $d$  is the viscous damping of the tool slide,  $F(s)$  is the transfer function for the controller,  $f_c$  is the cutting power.

- How large, in steady state, is the error  $e(t)$  between the actual value of the lathe tool and its desired reference value, when there is a step disturbance in the cutting power  $f_c(t)$ ? The controller is assumed to be an amplifier with a constant gain  $F(s) = K$ .
- How is this error changed if the amplifier is replaced by a PI controller with transfer function  $F(s) = K_1 + K_2/s$ ?

3.4 Consider the system

$$Y(s) = G(s)U(s) = \frac{0.2}{(s^2 + s + 1)(s + 0.2)}U(s).$$

- Suppose  $G(s)$  is controlled by a proportional controller with gain  $K_P$ , that is,

$$U(s) = K_P(R(s) - Y(s)).$$

Use MATLAB to compute the closed loop system, and to plot the step response of the closed loop system. Choose some values for  $K_P$  in the range 0.1 to 10. How are the properties of the step response affected by  $K_P$ ? What happens with the steady state error when  $K_P$  increases? Is it possible to obtain a well damped closed loop system and small steady state error using proportional control?

- Let us now introduce integration in the regulator and use

$$U(s) = (K_P + K_I \frac{1}{s})(R(s) - Y(s)).$$

Put  $K_P = 1$  and try some values of  $K_I$  in the range  $0 < K_I < 2$ . How are the step response and the steady state error affected by the introduction of the integrating part and the value of  $K_I$ ?

- Finally we will introduce the differentiating part in the regulator and use

$$U(s) = (K_P + K_I \frac{1}{s} + \frac{K_D s}{sT + 1})(R(s) - Y(s)).$$

Since true differentiation is difficult to implement, the derivative part is approximated by  $\frac{K_D s}{1 + sT}$ . (This will low-pass filter the error signal before differentiation.) Put  $K_P = 1, K_I = 1$  and  $T = 0.1$  and try some values of  $K_D$  in the range  $0 < K_D < 3$ . How does the D-part affect the step response of the closed loop system?

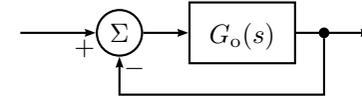


Figure 3.5a

- Draw a root locus with respect to  $K$  for the system in Figure 3.5a, with  $G_o(s)$  given below. For which values of  $K$  are the systems stable? What conclusions on the principal shape of the step response can be drawn from the root locus?

a) A Ferris wheel (Swedish: *Pariserhjul*):

$$G_o(s) = \frac{K(s+2)}{s(s+1)(s+3)}$$

b) A Mars rover:

$$G_o(s) = \frac{K}{s(s^2+2s+2)}$$

c) A magnetic floater:

$$G_o(s) = \frac{K(s+1)}{s(s-1)(s+6)}$$

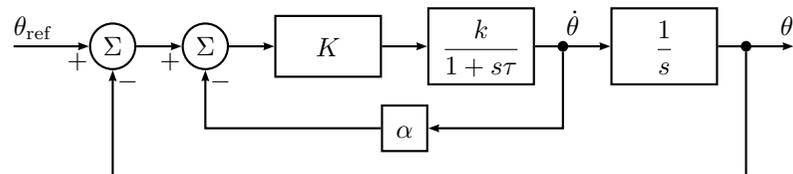


Figure 3.6a

3.6 Consider the servo system in Figure 3.6a with a DC-motor. Suppose that the angular velocity can be measured with a tachometer and let the control law be as in the block diagram. Let  $\tau = 0.5$  and  $k = 2$ .

- Draw the root locus with respect to  $K$  for the system without the tachometer feedback (that is,  $\alpha = 0$ ).
- Draw the root locus with respect to  $K$  for  $\alpha = 1$ .
- Draw the root locus with respect to  $K$  for  $\alpha = 1/3$ .
- Let  $K = 1$  and draw the root locus with respect to  $\alpha$ .

Discuss, using the results from a), b), c), and d), what is gained by using the tachometer.

3.7 Consider an aircraft where the pitch angle  $\theta$  is controlled by the elevator deflection (Swedish: *höjdroderutslag*)  $\delta$ , see Figure 3.7a. Let  $\omega$  be the angular velocity,

$$\omega = \dot{\theta}.$$

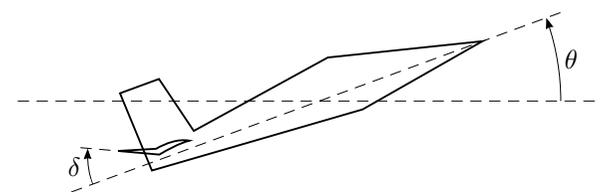


Figure 3.7a

If we consider small deviations from a reference value  $\theta_0$ , we get the transfer function from  $\delta$  to  $\omega$  for a specific aircraft as

$$G_2(s) = \frac{s+1}{(s+4)(s-3)}$$

This model is valid when the aircraft is flown with a large  $\theta_0$ . The elevator (Swedish: *höjdroder*) is driven by a hydraulic servo amplifier with the transfer function

$$G_1(s) = \frac{10}{s+10}$$

from elevator command  $\delta_{\text{ref}}$  to  $\delta$ .

- What happens with  $\omega$  if one gives a constant elevator command  $\delta_{\text{ref}}$ ? Motivate!
- The angular velocity  $\omega$  is measured and a control law is used so that the input  $\delta_{\text{ref}}$  to the servo amplifier is

$$K(\omega_{\text{ref}} - \omega).$$

Draw root locus with respect to  $K$ . For which values of  $K$  is the system stable?

- Is there any value of  $K$  such that the closed loop system is stable and all poles are real?

3.8 The block diagram in Figure 3.8a shows a cascade controlled DC-motor where  $K_1 > 0$  and  $K_2 > 0$ .

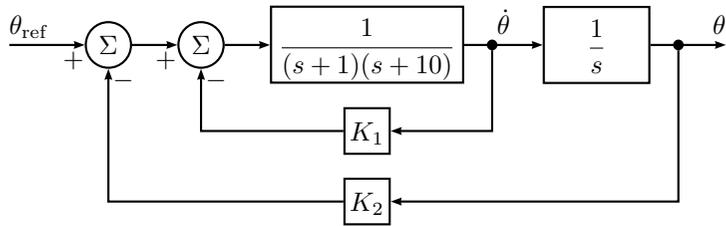


Figure 3.8a

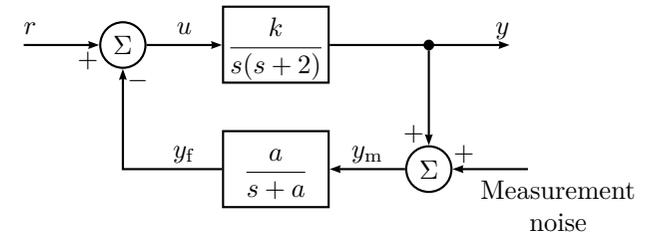


Figure 3.10a

- Draw root locus with respect to  $K_2$  for the characteristic equation of the closed loop system. For which  $K_2 > 0$  are the closed loop system asymptotically stable?
- How is the stability requirement on  $K_2$  affected by the size of the velocity feedback  $K_1$ ?

3.9 We want to control the temperature of an unstable chemical reactor. The transfer function is

$$\frac{1}{(s+1)(s-1)(s+5)}$$

- Use a proportional controller and draw a root locus with respect to the amplification  $K$ . Calculate which  $K$  in the compensator that stabilizes the system.
- Use a PD controller. The control law is given by

$$u = K(e + T_D \frac{de}{dt})$$

where  $e$  is the error. Let  $T_D = 0.5$  and draw a root locus with respect to  $K$ . For which values of  $K$  does the controller stabilize the system?

3.10 Consider the system in Figure 3.10a. In a realistic situation what you really measure is not  $y(t)$  but a signal  $y_m(t)$  which is the sum of  $y(t)$  and measurement noise. To avoid that the control is based on noisy measurements one uses  $y_f(t)$  instead of  $y_m(t)$ . The signal  $y_f(t)$  is  $y_m(t)$  filtered through the low pass filter

$$\frac{a}{s+a}$$

- First we assume that the measurement noise is negligible. Choose  $k = 6$ . Draw a root locus for the closed loop system with respect to the time constant of the low pass filter  $1/a$ . Find for which  $a > 0$  the system is stable.

- Use  $k = 6$  and assume that the noise is a high frequency sinusoid. The amplitude of  $y_f$  when

$$y_m(t) = \sin(10t)$$

is used as a measurement of how effective the noise reduction is. What is the smallest value you can obtain (after transients) by choosing a suitable  $a$ . We also want  $y$  to tend to the steady state value 1 when  $r(t)$  is a unit step.

3.11 Figure 3.11a shows the root locus for the characteristic equation of a P-controlled process  $G$  with respect to the gain  $K$ . In Figure 3.11b four step responses for the closed loop system with different values of  $K$  are shown.

Match the plots in Figure 3.11b with the  $K$ -values below. Justify your answer.

$$K = 4 \quad K = 10 \quad K = 18 \quad K = 50$$

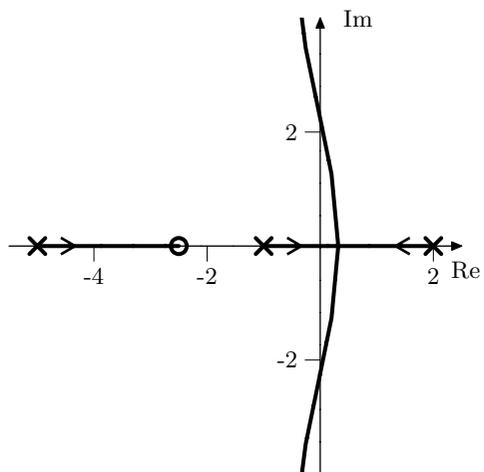


Figure 3.11a. Starting points are marked  $\times$  and end points  $\circ$ .

3.12 Consider a system with the transfer function

$$G(s) = \frac{s^{n-1} + b_1 s^{n-2} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

that has all zeros strictly in the left half plane. Show that such a system always can be stabilized by

$$u(t) = -Ky(t)$$

if  $K$  is selected large enough.

3.13 We want to control the level in a lake by controlling the flow using flood gates, see Figure 3.13a. The relationship between changes in the level of the lake,  $h_\Delta$ , and changes in the flows,  $q_{in,\Delta}$  and  $q_{out,\Delta}$ , is given by

$$\frac{d}{dt}(Ah_\Delta) = q_{in,\Delta} - q_{out,\Delta}$$

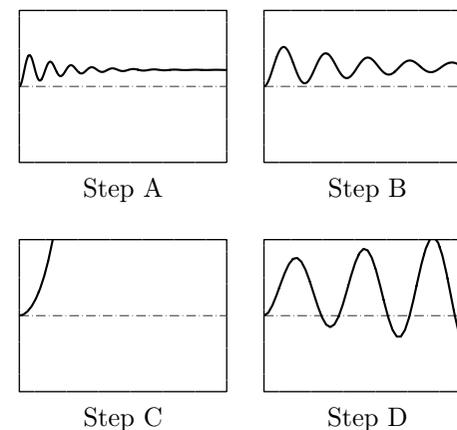


Figure 3.11b. All comparable axes have equal scaling.

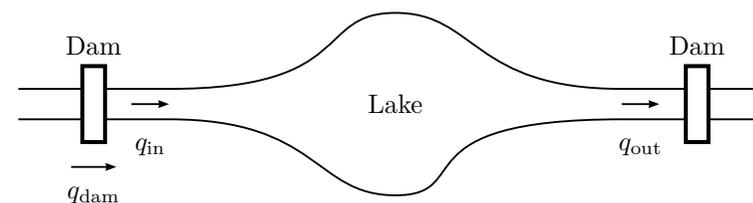


Figure 3.13a

where  $A$  is the area of the lake. In order to try to keep the level of the lake constant the flows through the lake are controlled at the inflow such that

$$q_{dam,\Delta} = K(h_{ref,\Delta} - h_\Delta)$$

where  $h_{ref,\Delta}$  is the reference value. The reservoir that controls the outflow is controlled so that  $q_{out}$  is constant, that is,  $q_{out,\Delta} = 0$ . Since it takes time before a change in  $q_{dam}$  gives a result in  $q_{in}$  we have

$$q_{in,\Delta} = q_{dam,\Delta}(t - T)$$

where  $T = 0.5$  hours. How large can the quotient  $K/A$  be at the most before the system becomes unstable?

3.14 A system  $G(s)$  is controlled using feedback with a proportional controller according to Figure 3.14a.

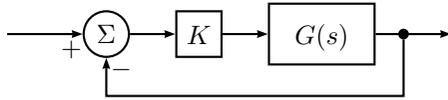


Figure 3.14a

- a) For  $K = 1$ , the open loop system  $KG(s)$  has the Nyquist diagram according to (i), (ii), (iii), or (iv) in Figure 3.14b. Is the closed loop system stable in each case?  $G(s)$  has no poles in the right half plane.
- b) If  $K > 0$ , for which values of  $K$  are the different closed loop systems stable?

- 3.15 a) Draw the Nyquist curve for an integrator  $G(s) = 1/s$ .
- b) Draw the Nyquist curve for the double integrator  $G(s) = 1/s^2$ .

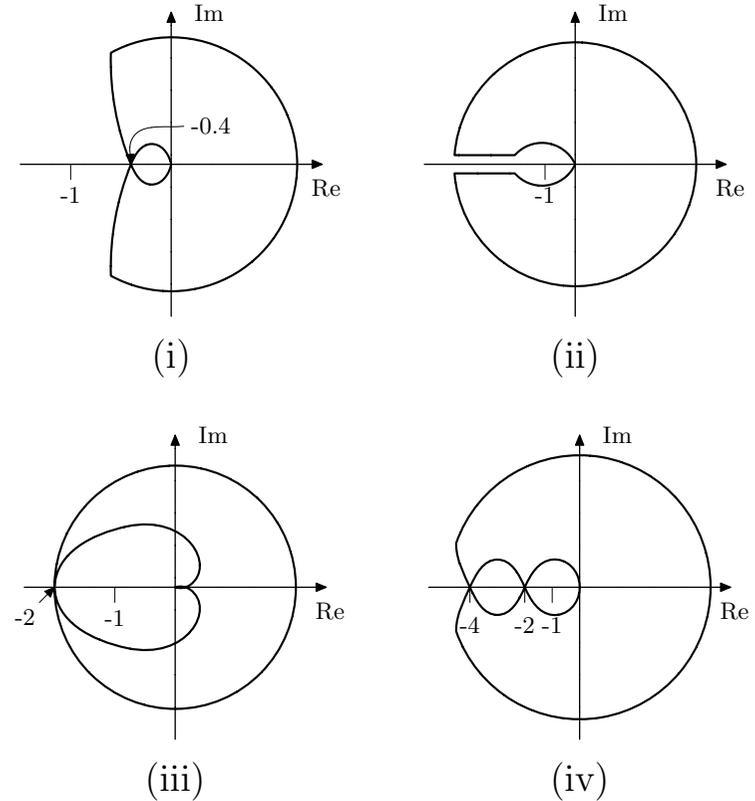


Figure 3.14b

- 3.16 The system  $G(s)$  is asymptotically stable and has the Nyquist curve in Figure 3.16a. It is controlled using feedback according to Figure 3.16b.
- a) For what values of  $K$  ( $K > 0$ ) is the closed loop system asymptotically stable?
- b) Determine the steady state error,  $e$ , as a function of  $K$  if  $y_{\text{ref}}$  is a unit step.
- c) Assume that  $G$  is controlled using an I controller according to Figure 3.16c. For what values of  $K$  is the closed loop system stable?

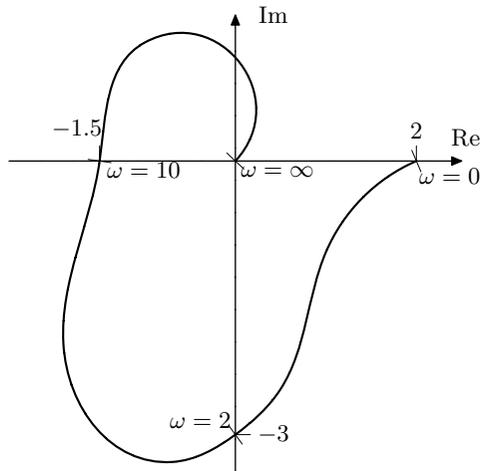


Figure 3.16a

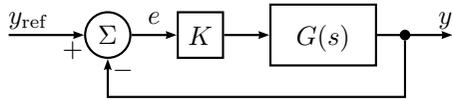


Figure 3.16b

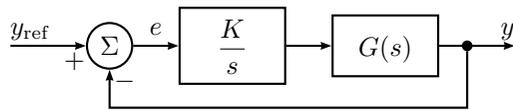


Figure 3.16c

3.17 Consider the DC-motor

$$\tau \ddot{y}(t) + \dot{y}(t) = u(t)$$

It is controlled by

$$u(t) = K(r(t - T) - y(t - T))$$

Here  $\tau$  and  $T$  are positive constants.  $K$  is slowly increased until the system oscillates with the angular frequency  $\omega = 1$ .  $K$  is then set to 33% of this value. After a while the system starts to oscillate again, now with the angular frequency  $\omega = 0.5$ . This is due to the fact that the time delay  $T$  has changed

to  $T_1$ . Can the parameters  $\tau$ ,  $T$ , and  $T_1$  be decided from these data? If so determine  $\tau$ ,  $T$  and  $T_1$ .

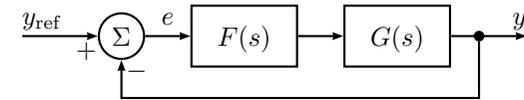


Figure 3.18a

3.18 A system  $G(s)$  is to be controlled using the regulator

$$F(s) = \frac{K}{s + 1}$$

according to the Figure 3.18a. The controller has positive gain  $K$  that, however, is not completely known. For what values of  $K$  is the closed loop system asymptotically stable? The Nyquist curve for  $G(s)$  is given in Figure 3.18b.

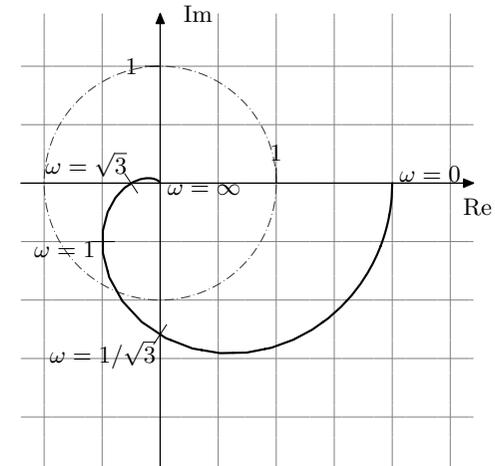


Figure 3.18b

- 3.19 The Nyquist curve for the system  $G(s) = \frac{B(s)}{A(s)}$  can be seen in Figure 3.19a. Determine which one of the root loci in Figure 3.19b that matches

$$A(s) + KB(s) = 0$$

for this system.

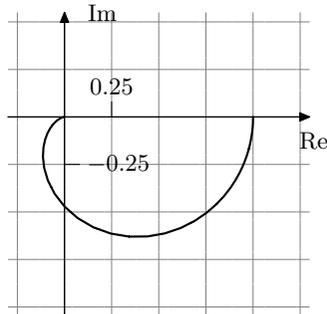


Figure 3.19a

- 3.20 The system  $\frac{1}{s(s+1)}$  with input  $u$  and output  $y$  has the controller

$$U(s) = F(s)(R(s) - Y(s)) \quad F(s) = \frac{b_0s^2 + b_1s + b_2}{s}$$

How should the coefficients of  $F$  be chosen to achieve pure P, pure I and pure D control respectively?

- 3.21 The equations for the P, PI, and PID controllers to be used in this problem are given in Problem 3.4.

a) Let the system

$$Y(s) = G(s)U(s) = \frac{0.2}{(s^2 + s + 1)(s + 0.2)}U(s)$$

be controlled by a proportional controller with gain  $K_P$ . Use MATLAB to plot the root locus with respect to  $K_P$  of the characteristic equation of the closed loop system. For which values of  $K_P > 0$  is the closed loop system asymptotically stable?

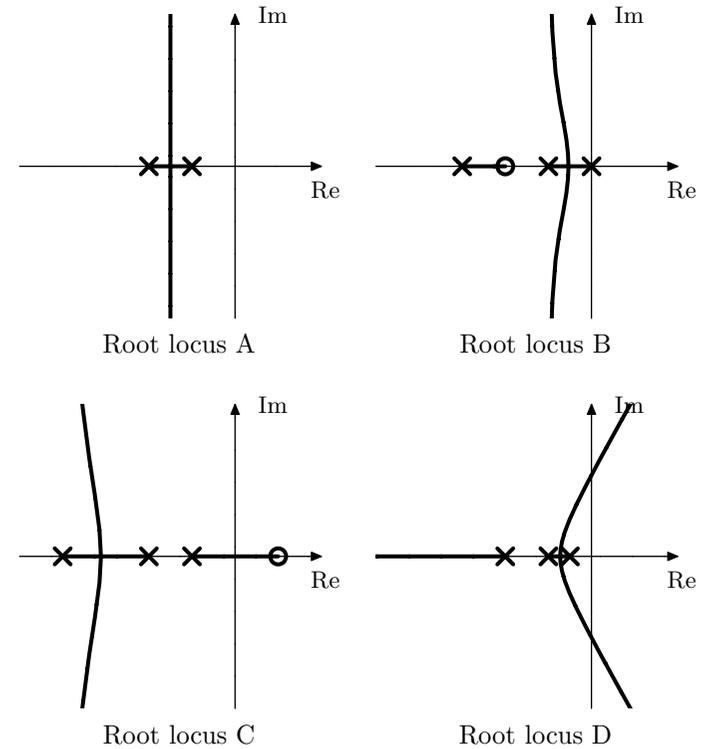


Figure 3.19b. Starting points ( $K = 0$ ) are marked  $\times$ , and end points ( $K \rightarrow \infty$ ) are marked  $\circ$ . All diagrams have equal scaling.

In Problem 3.4, we found that the step response was slow but well damped for small values of  $K_P$ , while it became faster but more oscillatory when  $K_P$  was increased. For large values of  $K_P$  the system became unstable. We also found that the steady state error was reduced when  $K_P$  was increased. Can these results be interpreted using the plot of the root locus?

- b) Now assume that the system is controlled by a PI controller where  $K_P = 1$ . Plot the root locus of the characteristic equation, with respect to  $K_I$ , and determine for which values of  $K_I > 0$  the closed loop system is asymptotically stable.

Problem 3.4 showed that an integrator eliminates the steady state error.

A small value of  $K_I$  gives a large settling time, while a too large value gives an oscillatory, and perhaps unstable closed loop system. Give an interpretation of these results using the root locus.

- c) Finally, let the system be controlled by a PID controller where  $K_P = 1$ ,  $K_I = 1$  and  $T = 0.1$ . Plot a root locus of the closed loop characteristic equation, with respect to  $K_D > 0$ , and relate the behavior of the root locus to the simulation result in Problem 3.4, that is, that the derivative part increases the damping of the closed loop system, but a too large  $K_D$  will give an oscillation with a higher frequency, and finally an unstable closed loop system.

3.22 Consider the system

$$Y(s) = G(s)U(s) = \frac{0.2}{(s^2 + s + 1)(s + 0.2)}U(s).$$

- a) Use MATLAB to plot the Nyquist curve of the open loop system when  $G(s)$  is controlled by a proportional regulator. Try some different values of  $K_P$  and find for which  $K_P$  the closed loop system is asymptotically stable. Compare your results with those from Problem 3.21a.
- b) Assume now that the system is controlled by a PI controller where  $K_P = 1$ . Investigate how  $K_I$  affects the Nyquist curve and determine for which values of  $K_I$  the closed loop system is asymptotically stable. Do you get the same results as in Problem 3.21b?
- c) Finally test a PID controller with  $K_P = 1$ ,  $K_I = 1$  and  $T = 0.1$  (cf Problem 3.4). How is the Nyquist curve affected by the value of  $K_D$ ?

3.23 a) Assume that the system

$$Y(s) = G(s)U(s) = \frac{0.4}{(s^2 + s + 1)(s + 0.2)}U(s)$$

is controlled by a proportional controller where  $K_P = 1$ . Use MATLAB to make a Bode plot of the open loop system and determine  $\omega_c$  (gain crossover frequency),  $\omega_p$  (phase crossover frequency),  $\varphi_m$  (phase margin) and  $A_m$  (gain margin) respectively. Compute the closed loop system and plot the step response.

- b) Now let  $K_P = 2.5$ . How does the change of  $K_P$  affect  $\omega_c$ ,  $\omega_p$ ,  $\varphi_m$ , and  $A_m$ ? Simulate the step response of the closed loop system and plot the result. How have the properties of the step response changed?

- c) How much can  $K_P$  be increased before the closed loop system becomes unstable? How does this value relate to the value of  $A_m$  that was obtained for  $K_P = 1$ ? Compute and plot the step response of the closed loop system for this value of  $K_P$ . How does the closed loop system behave in this case?

3.24 A system is controlled by a PID controller,

$$U(s) = (K_P + K_I \frac{1}{s} + K_D s)E(s)$$

In Figure 3.24a four step responses from a unit step for the parameter triples

- i)*  $K_P = 1$   $K_I = 0$   $K_D = 0$   
*ii)*  $K_P = 1$   $K_I = 1$   $K_D = 0$   
*iii)*  $K_P = 1$   $K_I = 0$   $K_D = 1$   
*iv)*  $K_P = 1$   $K_I = 1$   $K_D = 1$

are shown. Match each one of the parameter triples to one of the step responses. Justify your answer!

3.25 Assume that a DC-motor of the type described in Problem 2.1 is controlled by a proportional controller, that is,  $u(t) = K_P(\theta_{\text{ref}} - \theta)$ .

- a) Write down a block diagram for the control system. Compute the closed loop transfer function and determine how the poles of the closed loop system depend on the control gain  $K_P$ . Discuss what this means for the behavior of the system for different values of  $K_P$ .
- b) Determine the transfer function from the reference signal to the error. Let the reference signal be a step and a ramp respectively and determine what the control error will be in steady state in these two cases.
- c) Let the controller be a PI controller. What will the steady state error be in this case if the reference signal is a ramp?

3.26 Determine the transfer function for the loop gain and the closed loop system for the control system given by the block diagram in Figure 3.26a.

3.27 Figure 3.27a shows a block diagram of a control system. Determine the transfer function

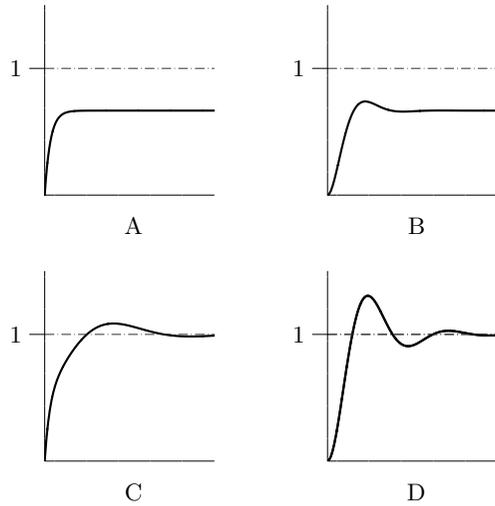


Figure 3.24a. Four step responses. All comparable axes have equal scaling.

Figure 3.24b

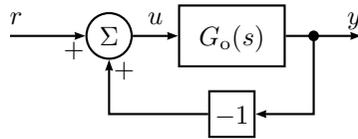


Figure 3.26a

- of the loop gain,
- of the closed loop system from  $R(s)$  to  $Y(s)$ ,
- from the disturbance  $N(s)$  to the output  $Y(s)$ ,
- from the reference signal  $R(s)$  to the error signal  $E(s)$ .

3.28 Consider again the control system in Figure 3.27a, with  $n = 0$  and

$$G(s) = \frac{1}{(s+1)(s+3)}$$

- Assume  $F(s) = K$ . Determine the steady state control error when  $r(t)$  is a step.

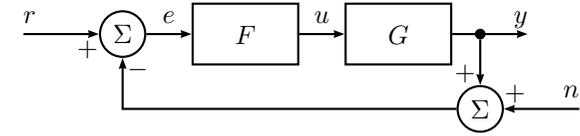


Figure 3.27a

- Determine a regulator  $F(s)$  such that the steady state error is zero when  $r(t)$  is a step.
- Assume  $F(s) = 1$ . Determine the poles and zeros of the closed loop system.

3.29 The system

$$Y(s) = \frac{1}{(s/0.6 + 1)(s + 1)}U(s)$$

is controlled using PID feedback

$$U(s) = (K_P + K_I \frac{1}{s} + K_D s)(R(s) - Y(s))$$

Figure 3.29a shows the step responses for the following four combinations of coefficient values. Combine the step responses and coefficients.

- $K_P = 4 \quad K_I = 0 \quad K_D = 0$
- $K_P = 4 \quad K_I = 3 \quad K_D = 0$
- $K_P = 4 \quad K_I = 1 \quad K_D = 0$
- $K_P = 4 \quad K_I = 0 \quad K_D = 4$

3.30 Continuously stirred tanks have an extensive use in chemical processes. They are often supplied with some sort of heat exchange system. If the tank is used for a chemical or biochemical reaction it is often important to keep a certain temperature to obtain desired productivity. A continuously stirred tank with a common type of heat exchange system is shown in Figure 3.30a. The tank is surrounded with a heating/cooling layer in which a liquid flows through in order to heat or chill the liquid in the tank.

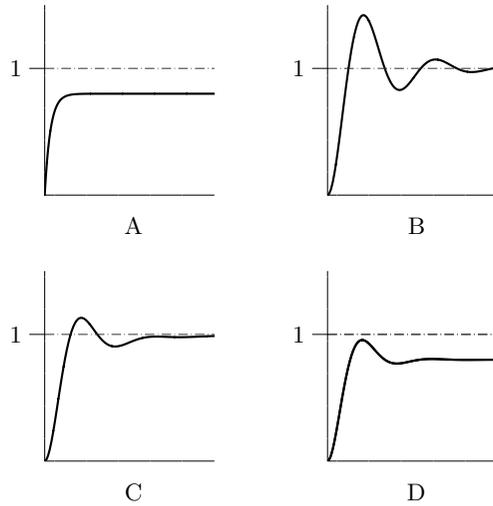


Figure 3.29a. Four step responses. All comparable axes have equal scaling.

- Determine the important signals of the system. Suggest a control strategy based on feedback and draw the block diagram.
- To be able to construct and evaluate different controllers it is necessary to have a process model. Determine a dynamical model for this system. Assume that the volumes in the tank and in the heat exchange system are constant.
- Linearize the model. (Stationary values:  $T_t^* = 50.0\text{ }^\circ\text{C}$ ,  $T_c^* = 75.0\text{ }^\circ\text{C}$ .)
- The parameters of the model are as follows

$$\begin{aligned} \rho_t = \rho_c = 1000.0\text{ kg/m}^3 \quad c_{t,\text{in}}^p = c_c^p = 4.2\text{ kJ/kg}^\circ\text{C} \quad U = 672\text{ kJ/}^\circ\text{Cmin} \\ F_t = 0.1\text{ m}^3/\text{min} \quad F_c = 0.2\text{ m}^3/\text{min} \quad V_t = 1.0\text{ m}^3 \quad V_c = 1.0\text{ m}^3 \\ T_{t,\text{in}} = 10.0\text{ }^\circ\text{C} \quad T_{c,\text{in}} = 95.0\text{ }^\circ\text{C} \end{aligned}$$

Determine the transfer function from flow in the heat exchange system to the temperature in the tank.

- Let the system be controlled by a P controller. Draw the root locus for the system.

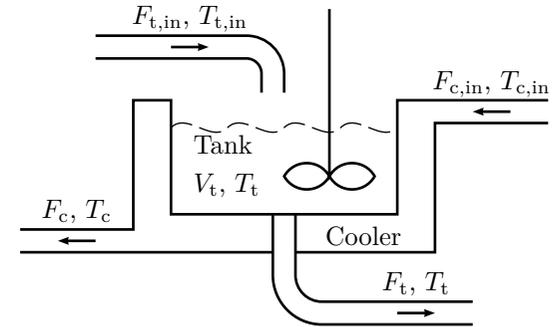


Figure 3.30a. Process consisting of a tank and a cooler. Input flows have a “in” subscript, while outputs have no such subscript since they are also the same as the quantities found inside the tank or cooler.

- 3.31 Consider a continuously stirred tank with a cooling system. In the tank a component A reacts to form component B in an exothermic reaction. This reaction is unstable, but possible to stabilize with feedback. A model for the purpose of control has been established

$$Y(s) = \frac{-1}{s^2 + 2s - 3} U(s)$$

where  $y(t)$  is the temperature in the tank and  $u(t)$  is the cooling flow.

- Show that the system is unstable.
- Prove that the system can be stabilized by a P controller.

- 3.32 Bacterial growth is described by the equation  $\dot{y} = \mu y$  where  $y$  is the amount of bacteria and  $\mu$  is a positive constant. Assume that we have a control signal  $u$  available that affects the speed of growth so that

$$\dot{y} = \mu y + u$$

One can then use a P controller  $u = K(r - y)$  where  $r$  is a reference signal. For which  $K$ -values will the system approach an equilibrium?

3.33 While working in space an astronaut has to be able to move. Necessary force is obtained by letting out gas from thrusters. For such a positioning control system the control law

$$u = K_1(r - y) - K_1K_2 \frac{dy}{dt}$$

is used, where  $u$  = thruster force,  $r$  = set point for the astronaut's position and  $y$  = the actual value of the astronaut's position. Draw a block diagram of the system, and use physics (Newton's law) to make a model of the astronaut. Also determine  $K_1$  and  $K_2$  such that

- If the set point  $r(t) = t$  there has to be a time  $T_0$  so that  $|r(t) - y(t)| < 1$  for  $t \geq T_0$ .
- The damping ratio of the closed loop system will be 0.7. The mass of the astronaut is assumed to be 100 kg, equipment included.

3.34 **2009-06-10 Upg. 2**

Givet systemet

$$G(s) = \frac{s + 2}{(s + 1)^2}$$

- a) Antag att systemet återkopplas med en P-regulator med förstärkning  $K$ ,  $0 \leq K < \infty$ . Rita rotort och ange för vilka värden på  $K$  som det återkopplade systemet är stabilt. Markera också i rotorten det val av poler som ger snabbast möjliga stegsvar utan svängningar.
- b) Antag att man istället använder en PI-regulator

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau) d\tau$$

Rita rotorten med avseende på  $K_I$ ,  $0 \leq K_I < \infty$ , då  $K_P = 4$ .

- c) Vilka kvalitativa skillnader finns det mellan stegsvaren för de återkopplade systemen i uppgift a) och b)?

## 4 Frequency Description

4.1 A mercury thermometer can be described with high accuracy as a first order linear time invariant dynamic system. The input is the real temperature and the output is the thermometer reading. In order to decide the transfer function in a thermometer it is placed in liquid where the temperature is varied as a sinusoid. The obtained result is shown in Figure 4.1a. Find the transfer function of the thermometer.

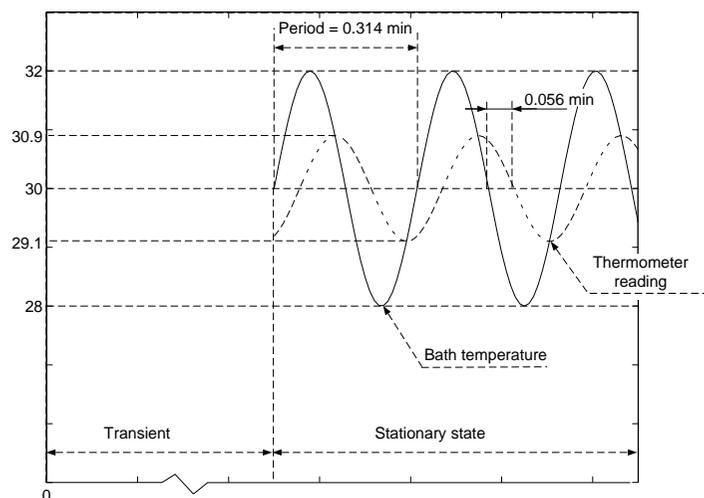


Figure 4.1a

4.2 We want to keep a ship on a given course,  $\Psi$ , with an automatic control system using the rudder angle  $\delta$ . See Figure 4.2a. If  $\omega$  denotes the angular velocity of the ship,

$$\omega = \dot{\Psi} \quad (4.1)$$

the following differential equation is valid for small values of  $\omega$  and  $\delta$ ,

$$T_1 \dot{\omega} = -\omega + K_1 \delta \quad (4.2)$$

where  $T_1 = 100$  and  $K_1 = 0.1$ . The desired course,  $\Psi_{\text{ref}}$ , and the measured course,  $\Psi$ , are fed in to the auto pilot, which gives the signal  $u$  to the rudder

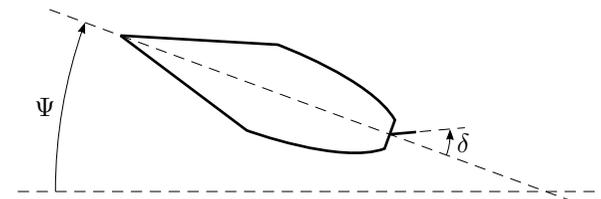


Figure 4.2a

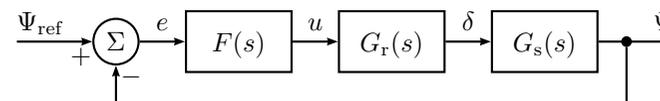


Figure 4.2b

engine. Figure 4.2b shows a block diagram of the auto pilot. The auto pilot has the transfer function

$$F(s) = K \frac{1 + \frac{s}{a}}{1 + \frac{s}{b}}, \quad a = 0.02, b = 0.05$$

while  $G_r$  is given by

$$G_r(s) = \frac{1}{1 + sT_2}, \quad T_2 = 10$$

and  $G_s(s)$  is defined by (4.1) and (4.2).

a) Make a Bode plot for the transfer function  $FG_rG_s$ , for  $K = 0.5$ .

- b) At the testing of the auto pilot we do the following experiment. The gain of the auto pilot  $K$  is increased until the control system oscillates with constant amplitude. At what value of  $K$  does this occur? What is the period time of the oscillation?
- c)  $\Psi_{\text{ref}}$  is allowed to vary as a sinusoid

$$\Psi_{\text{ref}}(t) = A \sin \alpha t$$

where  $A = 5^\circ$  and  $\alpha = 0.02$ . When the movements of the ship have stabilized we have

$$\Psi(t) = B \sin(\beta t + \varphi)$$

What values do  $B$ ,  $\beta$ , and  $\varphi$  have if  $K = 0.5$ ?

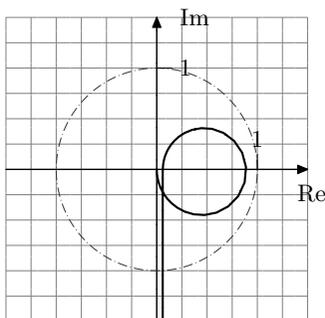


Figure 4.3a

- 4.3 a) In Figure 4.3a the Nyquist curve for a system is shown. Draw the Bode plot for the same system. The scale on the  $\omega$ -axis is not important, as long as the amplitude and phase curve are in agreement.
- b) Draw a diagram for the poles and zeros for the system. The relative placement is important, not the scale.

- 4.4 Figure 4.4a shows the step responses (when the input is a unit step) and Bode gain plots of four different systems, in no particular order. Identify the pair of plots that belongs to each system. That is, for each step response, find the corresponding Bode gain plot (amplitude curve). Motivate your answer by pointing out a set of unique features for each system.

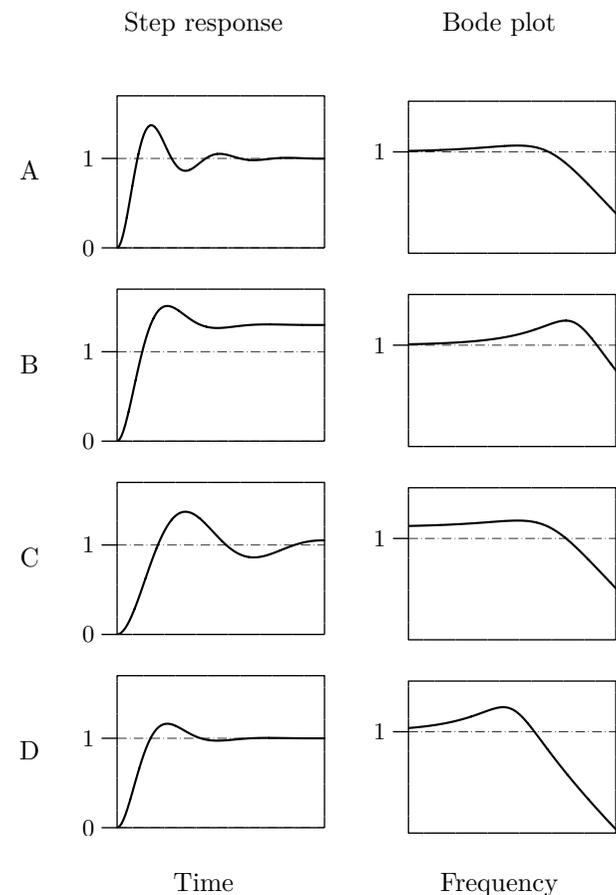


Figure 4.4a. All comparable diagrams have equal scaling.

- 4.5 a) Consider the transfer functions  $G_A(s)$ ,  $G_B(s)$ ,  $G_C(s)$ ,  $G_D(s)$ , and  $G_E(s)$  in Problem 2.6.

$$G_A(s) = \frac{1}{s^2 + 2s + 1}, \quad G_B(s) = \frac{1}{s^2 + 0.4s + 1}$$

$$G_C(s) = \frac{1}{s^2 + 5s + 1}, \quad G_D(s) = \frac{1}{s^2 + s + 1}$$

$$G_E(s) = \frac{4}{s^2 + 2s + 4}$$

Study the amplitude curves of the Bode plots for the systems and find the static gain and bandwidth of the systems. In cases when it is relevant find also the resonance frequency and resonance peak.

- b) Describe qualitatively (without formulas) the relationships between  $T_r$  (rise time) and  $\omega_B$  (bandwidth) and between  $M$  (overshoot) and  $M_p$  (resonance peak) respectively?

- 4.6 A system has the transfer function

$$G(s) = \frac{e^{-2s}}{s(s+1)}$$

What is the output (after transients) when the input is

$$u(t) = 2 \sin(2t - 1/2)$$

- 4.7 For the systems below the input is chosen as  $u(t) = \sin(2t)$ . Determine the output signal  $y(t)$  after transients have faded away, provided that it exists.

- a)  $Y(s) = \frac{1}{s+1}U(s)$   
 b)  $Y(s) = \frac{1}{s-1}U(s)$   
 c)  $Y(s) = \frac{1}{(s+1)(2s+1)}U(s)$   
 d)  $Y(s) = \frac{e^{-0.5s}}{s+1}U(s)$

- 4.8 A system is described by  $Y(s) = G(s)U(s)$ . Figure 4.8a shows  $u(t) = \sin(\omega t)$  and the corresponding output  $y(t)$  (after all transients have faded away) for the frequencies  $\omega = 1, 5, 10,$  and  $20$  rad/s (from top to bottom).

- a) Determine the gain ( $|G(i\omega)|$ ) and phase ( $\arg G(i\omega)$ ) for the system for each value of  $\omega$ .  
 b) Determine the gain values in dB<sub>20</sub> ( $20 \log_{10}(|G(i\omega)|)$ ).  
 c) Sketch the Bode plot using the values determined above.

- 4.9 Combine the transfer functions below with the Bode plots in Figure 4.9a.

$$G_1(s) = \frac{1}{s+1}, \quad G_2(s) = \frac{6(s+1)}{(s+2)(s+3)}$$

$$G_3(s) = \frac{1}{s}, \quad G_4(s) = \frac{1}{s(s+1)}$$

$$G_5(s) = \frac{5}{s^2 + 2s + 5} \text{ (poles: } -1 \pm i2)$$

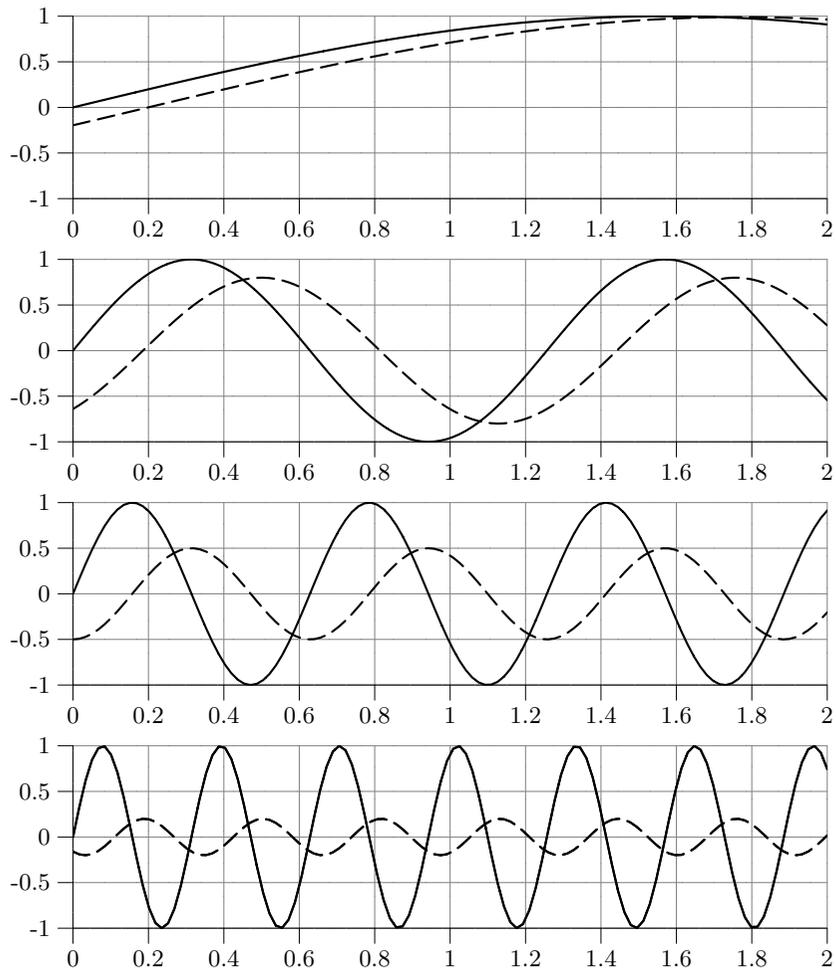


Figure 4.8a.  $u(t) = \sin(\omega t)$  (solid) and  $y(t)$  (dashed).

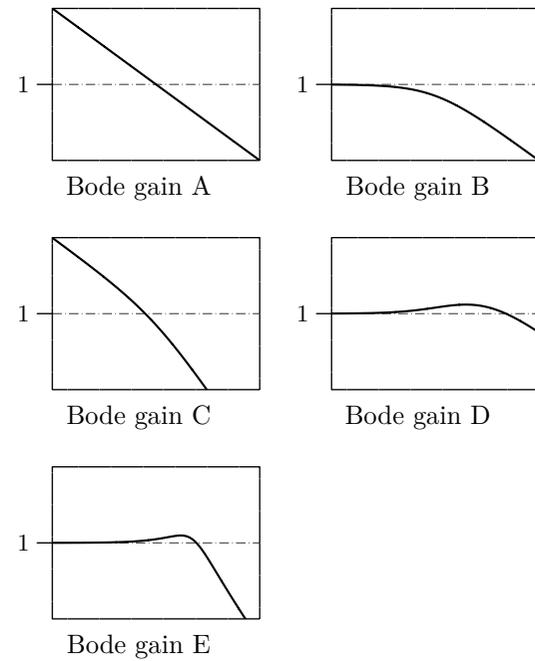


Figure 4.9a. All diagrams have equal scaling.

4.10 Figure 4.10a shows the Bode gain plots and step responses of four different systems, in no particular order. Identify the pair of plots that belongs to each system. That is, for each Bode gain plot (amplitude curve), find the corresponding step response. Motivate your answer by pointing out a set of unique features for each system.

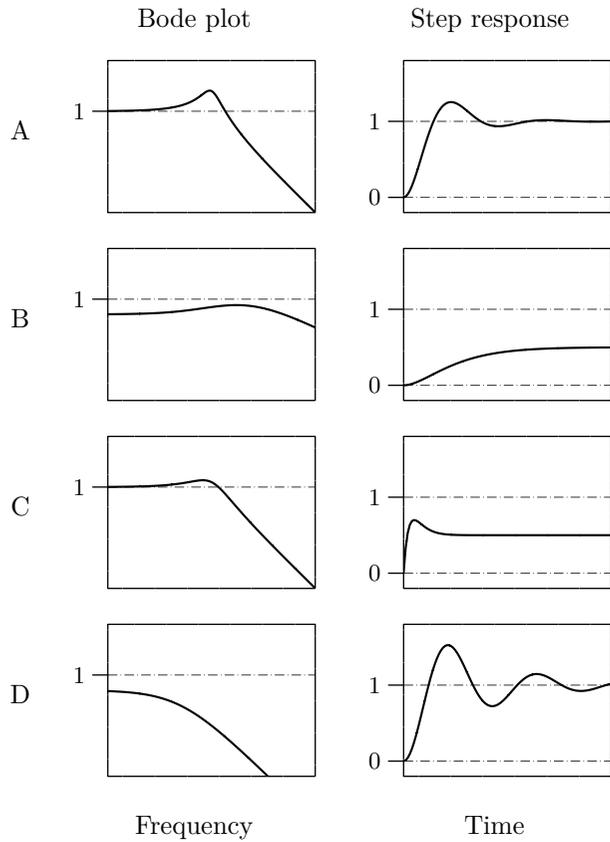


Figure 4.10a. All comparable diagrams have equal scaling.

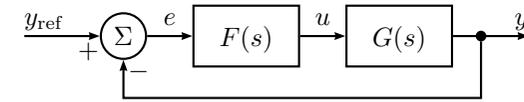


Figure 4.11a

4.11 The pH in a biochemical reactor is controlled by addition of a base. The transfer function  $G(s)$  from added base to pH for the open system has been determined by experiments to be

$$G(s) = \frac{1.7}{(s+1)(0.7s+1)(0.5s+1)}$$

In an attempt to control the pH the control structure shown in Figure 4.11a is employed

- Make a Bode plot for the transfer function  $G(s)$ .
- Assume that a P controller is used ( $F(s) = K$ ). At what value of  $K$  does the pH start to oscillate with constant amplitude?

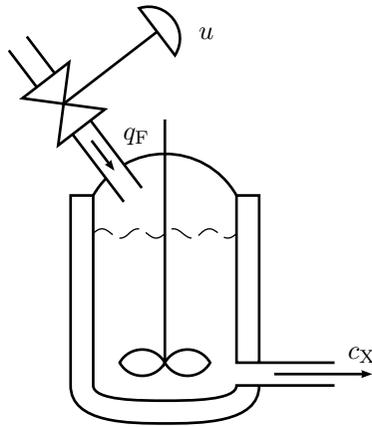


Figure 4.12a

4.12 Consider the biochemical reactor in Figure 4.12a. It is desirable to control the concentration of biochemical material  $c_X$  (output  $y$ ) by manipulating the dilute flow  $q_F$  (input  $u$ ). A model of this system can be described as

$$Y(s) = \frac{2e^{-5s}}{30s + 1}U(s)$$

where the time delay reflects the time it takes to measure the biochemical concentration. The bode diagram of the system is shown in Figure 4.12b.

- For which values of  $K$  is a P controller going to stabilize the system?
- Construct a controller which has the crossover frequency  $\omega_{c,d} = 0.1$  and no steady-state error.

4.13 A system

$$Y(s) = G_o(s)U(s)$$

is used with a controller

$$U(s) = R(s) - Y(s).$$

- In Figure 4.13a are Bode diagrams of the open loop system  $G_o$  for two different systems (top), the closed loop systems  $G_c$  (second row), step

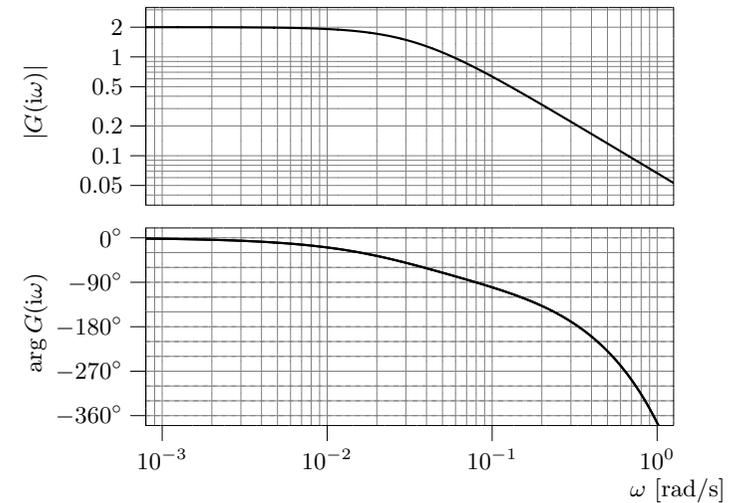


Figure 4.12b

responses of the closed loop systems (third row) and the poles of the closed loop systems (bottom). Assign each closed loop system, closed loop step response and closed loop pole diagram to the correct open loop system.

- Solve the same task for Figure 4.13b.

4.14 **2010-10-19 Upg. 1a**

Då man applicerar insignalen

$$u(t) = \sin(2t), \quad t \geq 0$$

på ett första ordningens system

$$\dot{y}(t) + ay(t) = bu(t), \quad y(0) = y_0$$

får man utsignalen

$$y(t) = 2 \sin(2t - \pi/4).$$

Bestäm systemparametrarna  $a$  och  $b$  samt initialvärdet  $y_0$ .

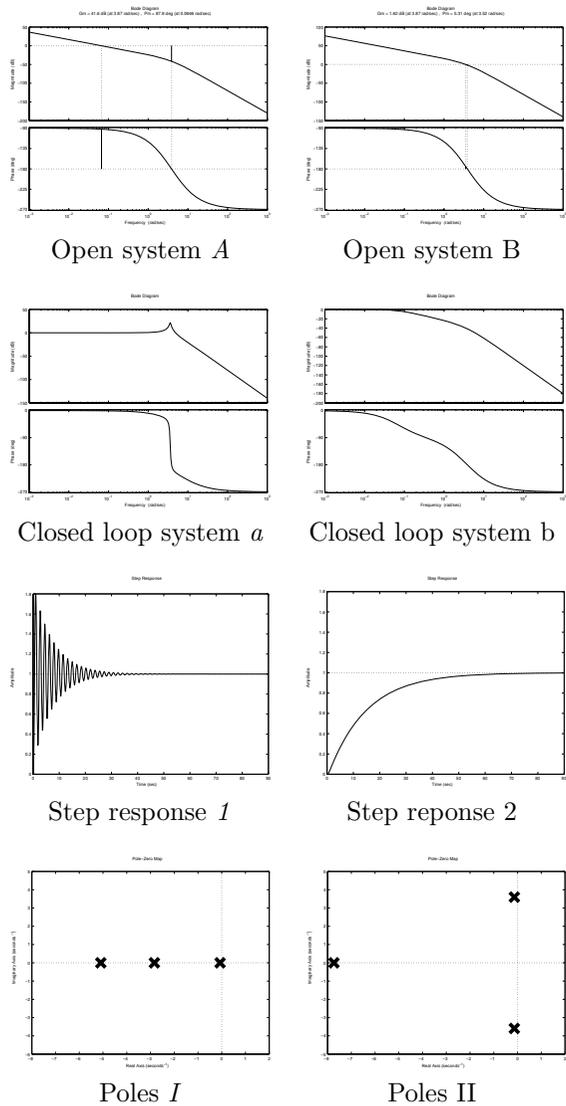


Figure 4.13a. The frequency scales of the Bode diagrams are the same. The time scales of the step responses are the same. In the pole-zero maps,  $\times$  marks poles.

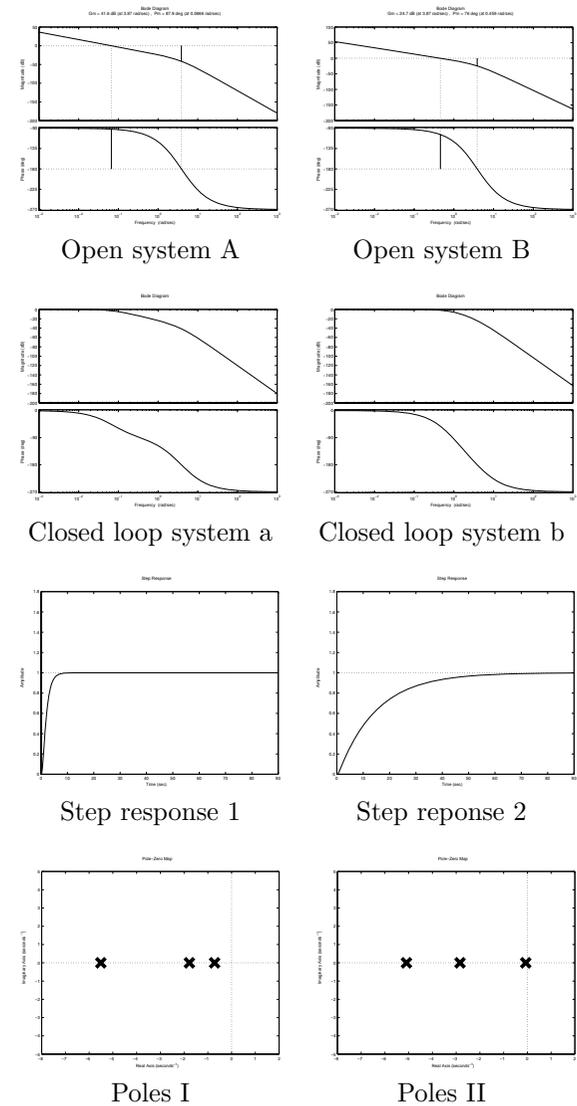


Figure 4.13b. The frequency scales of the Bode diagrams are the same. The time scales of the step responses are the same. In the pole-zero maps,  $\times$  marks poles.

# 5 Compensation

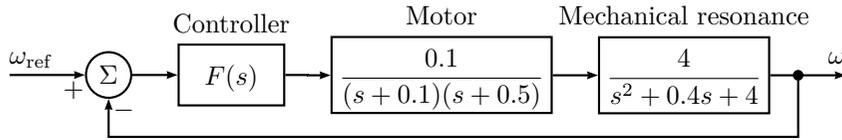


Figure 5.1a

5.1 The block diagram for speed control of a DC-motor is shown in Figure 5.1a. Find a compensator  $F(s)$  such that the following specifications are fulfilled.

- The system should be twice as fast as for  $F(s) = 1$ , but with the same damping as for  $F(s) = 1$ .
- If  $\omega_{\text{ref}}$  is constant,  $|\omega_{\text{ref}} - \omega| / \omega_{\text{ref}}$  should be less than 5%.
- The controller should not be unnecessarily sensitive for high frequency disturbances and the open loop system's low frequency amplification should not be larger than necessary.

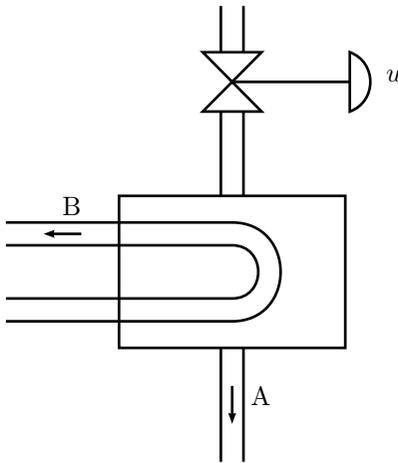


Figure 5.2a

5.2 The outflow temperature  $\theta$  in the liquid A can be controlled in a heat exchanger by controlling the flow of the liquid B by a valve with the setting denoted  $u$ .

See Figure 5.2a. Measurements have been made using a sinusoidal input  $u$  and the gain and phase shift have been measured at different frequencies. The results are given in the following table.

Frequency [rad/s]	Gain	Phase shift
0.05	1.37	$-67^\circ$
0.1	0.80	$-106^\circ$
0.2	0.34	$-153^\circ$
0.3	0.18	$-185^\circ$
0.4	0.11	$-210^\circ$

- Make a Bode plot for the system.
- What is the largest crossover frequency possible to achieve when using proportional control and wanting a phase margin of at least  $50^\circ$ ?
- Suggest a compensator that doubles the speed compared to b) and still keeps the phase margin.

5.3 A hydraulic system with a valve and a piston is described by the following linearized transfer function

$$G(s) = \frac{k_u/A}{s(\frac{s^2}{\omega_0^2} + 2\zeta\frac{s}{\omega_0} + 1)}$$

where  $A$  is the area of the piston and  $k_u$  the hydraulic gain.

- Make a Bode plot for the system when  $\omega_0 = 150$  rad/s,  $\zeta = 0.1$ , and  $k_u/A = 20$ .
- What is the smallest value of the ramp error that can be achieved using proportional control if we want an amplitude margin of 2? What is the crossover frequency in this case?
- Suggest a compensator such that the ramp error decreases 15 times at the same time as the crossover frequency, phase margin and amplitude margin will be the same as in b). Due to physical constraints in the implementation, this regulator has to have finite amplification at all frequencies.

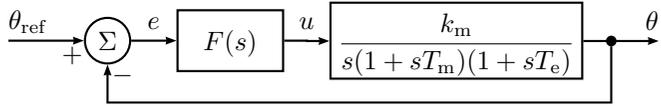


Figure 5.4a

5.4 Figure 5.4a shows a position servo including a DC-motor. The extra time constant  $T_e$  is due to the inductance in the winding of the motor, which is usually not taken into account. The parameter values are  $k_m = 10$ ,  $T_m = 0.1$  and  $T_e = 0.01$ . We want the servo to fulfill the following specifications:

- Rise time  $\leq 0.1$  s.
- Overshoot  $\leq 10\%$ .
- The steady state error at step in  $\theta_{\text{ref}}$  should be zero.
- The steady state error when  $\theta_{\text{ref}}$  is a ramp with slope  $10^\circ/\text{s}$  should be less than  $0.1^\circ$ .

Suggest a compensator such that the specifications are fulfilled. (Clue: Suppose that the relationship between rise time, overshoot, and other specifications are the same as for a second order system, that is, according to Figures 5.4b and 5.4c in Solution 5.4 (the figures can also be found in Glad&Ljung).)

5.5 In Figure 5.5a we have arranged step responses and open loop and closed loop Bode plots for five different systems. Identify the three plots that belong to each of the five systems, one open loop and one closed loop Bode plot and one step response. Motivate your answer by pointing out one unique feature for each system.

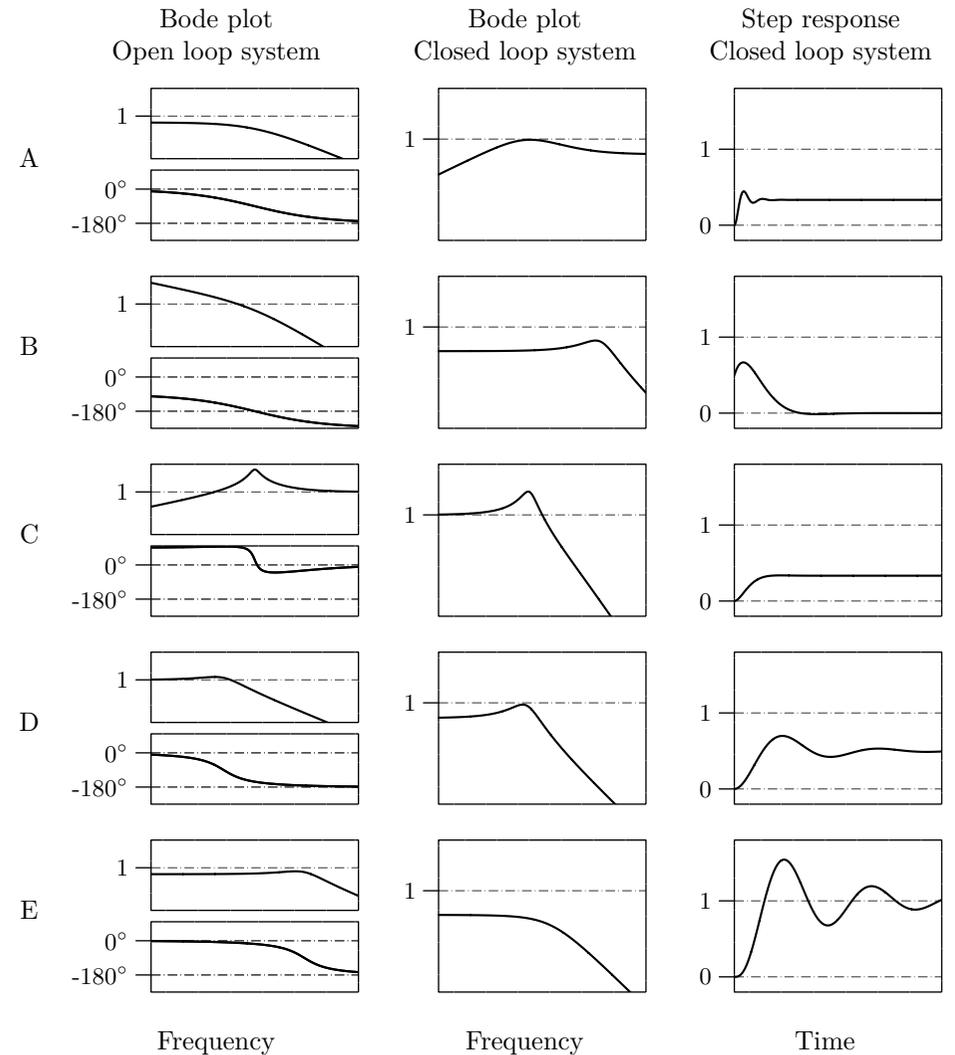


Figure 5.5a. All comparable diagrams have equal scaling.

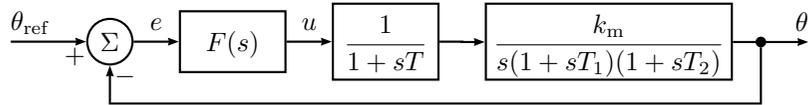


Figure 5.6a

5.6 A DC-servo is described by the block diagram in Figure 5.6a, where  $T_1 = 50$  ms is a mechanical time constant,  $k_m = 10$  is a proportional constant,  $T_2 = 25$  ms is an electrical time constant, and  $T = 10$  ms is an amplifier time constant. The system is tested with  $F(s) = 1$  and we find that the dynamic properties are satisfactory but that the system is somewhat too slow. Find an  $F(s)$  so that the closed loop system is twice as fast as for  $F(s) = 1$ , without increasing the overshoot.  $F(s)$  should also give a closed loop system which fulfills the following accuracy demands:

- $|\theta - \theta_{\text{ref}}| \leq 0.001$  rad in steady state when  $\theta_{\text{ref}}$  is constant.
- When  $\theta_{\text{ref}}$  is a ramp with slope 10 rad/s we should have  $|\theta - \theta_{\text{ref}}| \leq 0.01$  rad in steady state.

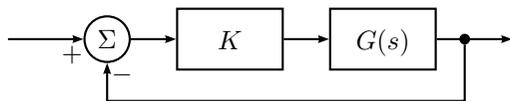


Figure 5.7a

5.7 The amplitude curves and the phase curves in Figure 5.7b have been measured for a system without poles in the right half plane. The system is controlled using feedback according to Figure 5.7a. Use the Nyquist criterion to decide for which values of  $K$  the closed loop system is stable ( $K > 0$ ).

5.8 A block diagram for a control system with time delay is shown in Figure 5.8a. The system  $G_1$  has no poles in the right half plane.

- $G_1$  has a Bode plot according to the plot in Figure 5.8b. Determine for what values of the time delay  $T$  the closed loop system is stable.
- The same as in a), but for the plot in Figure 5.8c.

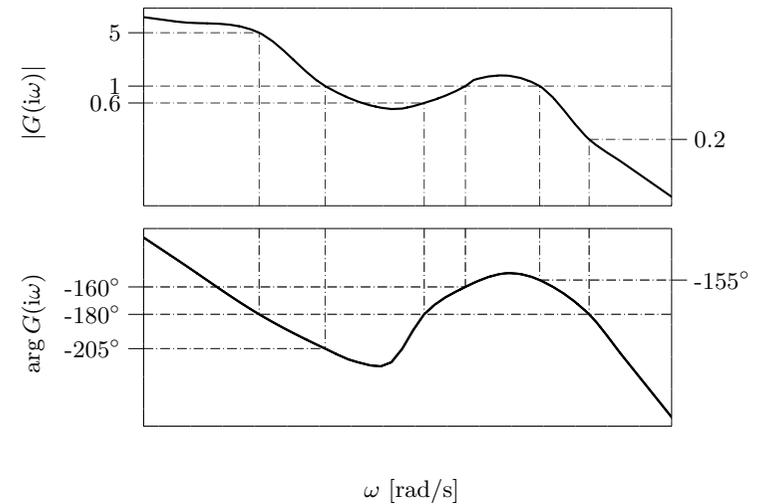


Figure 5.7b

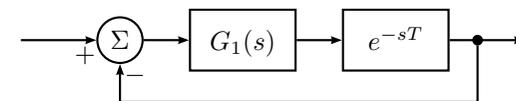


Figure 5.8a

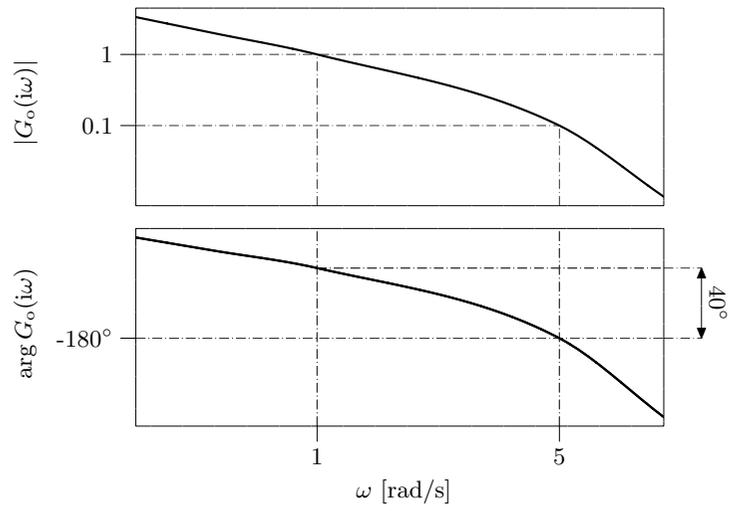


Figure 5.8b

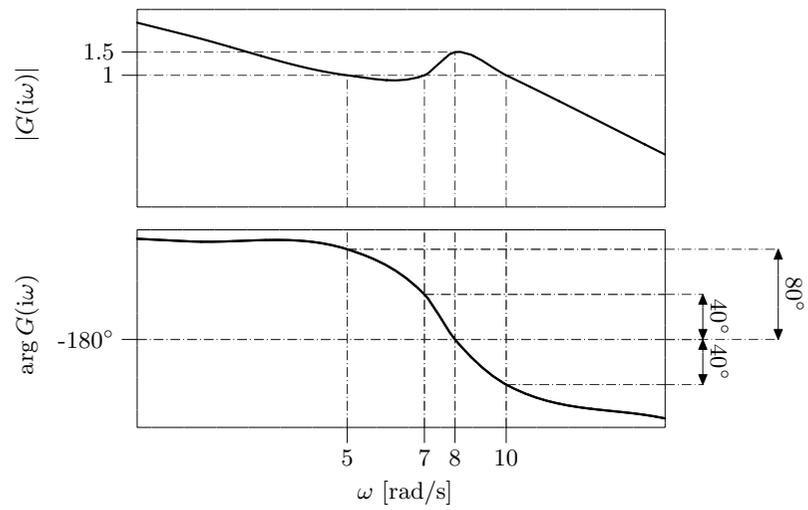


Figure 5.8c

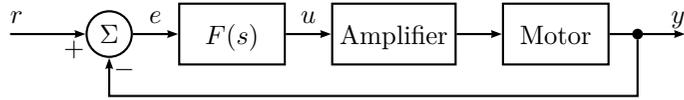


Figure 5.9a

5.9 A servo system based on a DC-motor has to be designed. A block diagram for the system is given in Figure 5.9a. By measuring the phase shift and the gain at different frequencies the Bode plot for the motor, see Figure 5.9b, has been determined. The amplification is a system of the first order, that is, it has the transfer function

$$G_A(s) = \frac{k_A}{s + a}.$$

In order to find the constants  $k_A$  and  $a$ , a unit step experiment has been carried out on the amplifier, giving the output shown in Figure 5.9c.

- a) Find the constants  $k_A$  and  $a$  from Figure 5.9c. Also draw the Bode plot for the open loop system, that is, the system from  $u$  to  $y$ .
- b) Find a compensator  $F(s)$ , such that the closed loop system fulfills the following demands:
  - ◇ The system has to be 5 times as fast as when using  $F(s) = 1$ .
  - ◇ The overshoot should not be larger than for  $F(s) = 1$ .

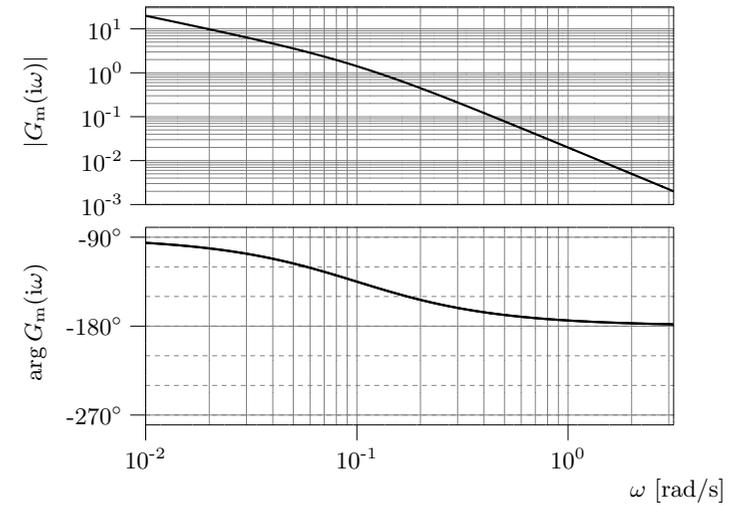


Figure 5.9b

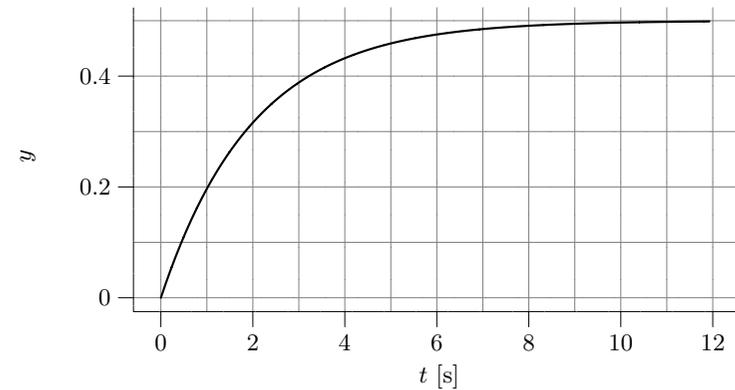


Figure 5.9c

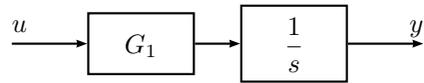


Figure 5.10a

5.10 A system  $G(s)$  can be split into two sub-systems

$$G(s) = G_1(s) \frac{1}{s}$$

according to Figure 5.10a. The Bode plot for  $G_1(s)$  is given in Figure 5.10b. Find a compensator for the system  $G(s)$  such that the following is fulfilled:

- The phase margin for the compensated system is  $40^\circ$ .
- The closed loop system is twice as fast as what is possible to achieve using proportional control with a  $40^\circ$  phase margin.
- The steady state error when the reference signal is a ramp is 1% of the corresponding error with proportional control and  $40^\circ$  phase margin.

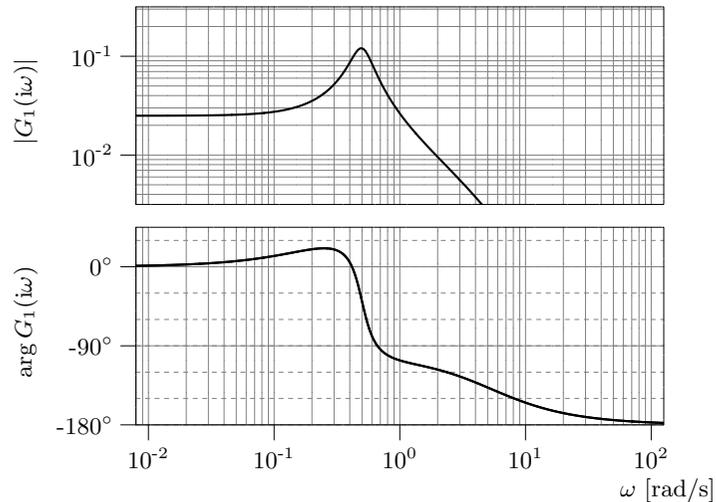


Figure 5.10b

5.11 The Bode plot for a system is given in Figure 5.11a.

- Draw the Nyquist curve of the system.
- Assume that the system is controlled using the proportional feedback

$$U(s) = K(R(s) - Y(s))$$

For which  $K > 0$  is the closed loop system asymptotically stable?

- Assume that we choose  $K = 2$  in the proportional controller in problem b). What will the steady state error be when  $r(t) = 10t$ ?
- Assume that  $y(t)$  is delayed  $T$  seconds. How large is  $T$  allowed to be in order for the system to still be asymptotically stable with  $K = 2$ ?

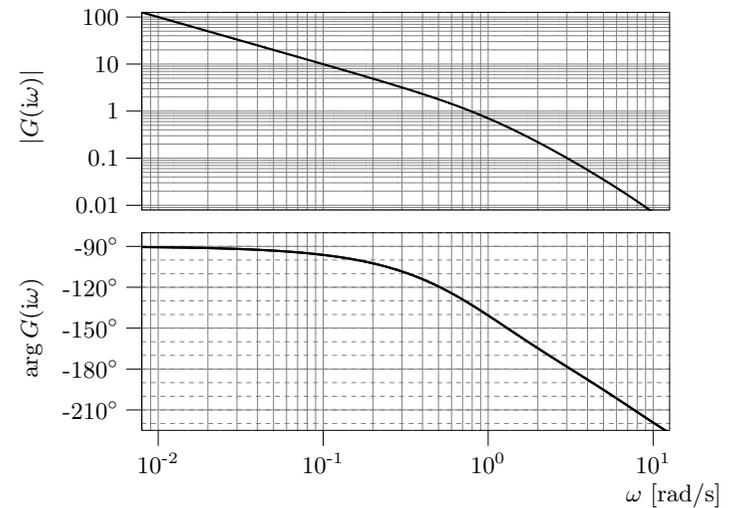


Figure 5.11a

5.12 a) A plot of the amplitude curve of a stable transfer function  $G_o(s)$  is given in Figure 5.12a. Choose one of the following alternatives regarding the stability of the closed loop system  $\frac{G_o}{1+G_o}$ :

1. It is stable.
2. It is not stable.
3. Impossible to determine given these facts only.

b) Repeat for the transfer function whose amplitude curve is given in Figure 5.12b. Justify your answers carefully.

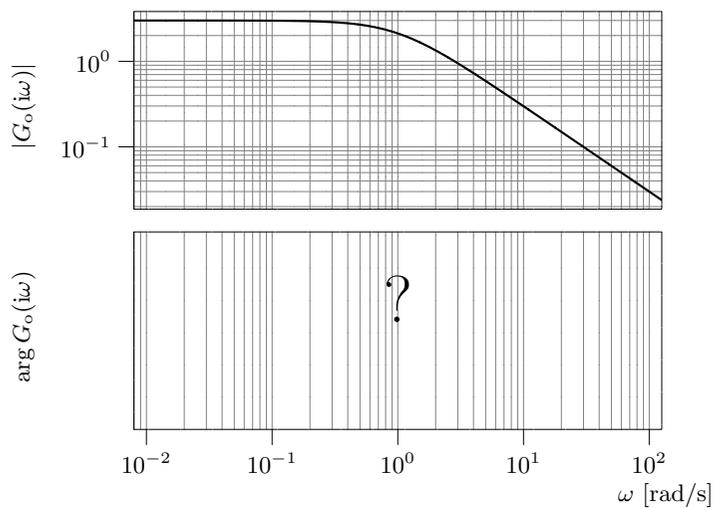


Figure 5.12a

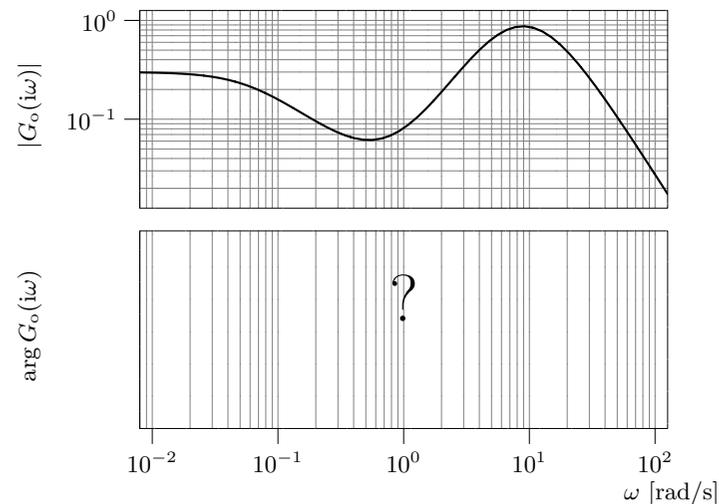


Figure 5.12b

5.13 Consider the relation

$$Y(s) = G(s)U(s)$$

where

$$G(s) = \frac{725}{(s+1)(s+2.5)(s+25)}$$

a) Assume that the system is controlled by

$$U(s) = F(s)(R(s) - Y(s))$$

where  $F(s) = 1$ . Find  $\omega_c$ ,  $\omega_p$ ,  $\varphi_m$ , and  $A_m$  for the loop gain.

b) Compute a regulator such that the open loop system fulfills the following requirements:

- (i)  $\omega_c = 5$
- (ii)  $\varphi_m \geq 60^\circ$

and the closed loop system fulfills:

- (iii)  $e_0 = 0$

Draw the Bode plot of the compensated open loop system and check that the requirements are satisfied. Simulate the closed loop system for a step in the reference signal and plot the step response. Check that the requirement on the steady state error is satisfied.

c) Draw the amplitude curve of the Bode plot of the closed loop system with and without the compensator. Describe how the properties of the closed loop system have been changed by the compensation.

d) Simulate the control error when the reference signal is a ramp and the regulator designed in b) is used. Is the stationary error zero?

5.14 When using microorganisms in production it is important to keep the oxygen concentration at a certain level to get maximum productivity. There are many ways to control the amount of dissolved oxygen, in this example we will use the speed of stirring as the controlled signal. The transfer function from the stirrer speed  $N_\Delta$  to the oxygen measurement  $O_{p\Delta}$  becomes (linearized model)

$$G(s) = \frac{b}{s + T_1} \frac{e^{-s\tau}}{1 + sT_2}$$

The parameters  $\tau = 2$  s,  $T_2 = 20$  s and  $b = 0.02$  remain constant with change in stirrer speed while  $T_1$  can vary from  $0.02$  s<sup>-1</sup> to  $0.224$  s<sup>-1</sup> as the stirrer speed increases from 400 r/min to 1200 r/min. A Bode plot for  $G(s)$  is given in Figure 5.14a. Construct a controller, for the 1200 r/min case, which has a crossover frequency  $\omega_c = 0.2$  rad/s, a phase margin  $\varphi_m = 60^\circ$  and no steady-state error.

5.15 Earlier in the history of the basic automatic control courses at LiTH, the lead and lag compensators were parameterized in a different way. This parameterization is used in this problem, which is not only meant to be a crash course in understanding exams with solutions dating back to those days, but should be

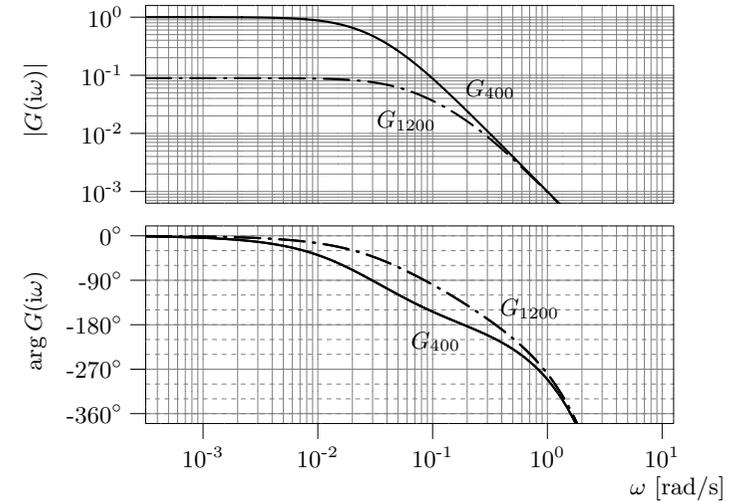


Figure 5.14a. Solid line: 400 r/min. Dash-dotted line: 1200 r/min.

seen as a chance of practicing the *ideas* of loop shaping rather than exercising *recipe* knowledge.

The old parameterization of a lead compensator looked as follows:

$$F_{\text{lead}}(s) = N \frac{s + b}{s + bN}$$

The schematic Bode plot of such a lead compensator is shown in Figure 5.15a, and it can be shown that the maximum phase lead is obtained at the frequency  $b\sqrt{N}$ , where the gain is  $\sqrt{N}$ , and the phase lead is given by

$$\varphi_{\text{max}} = \arctan \frac{\sqrt{N} - 1/\sqrt{N}}{2}$$

(shown in Figure 5.15b).

The old parameterization of a lag compensator looked as follows:

$$F_{\text{lag}}(s) = \frac{s + a}{s + a/M}$$

The schematic Bode plot of such a lag compensator is shown in Figure 5.15c. Just like the parameter  $\tau_1$  of the new parameterization, the parameter  $a$  used

here shall be determined as a trade-off between the undesirable phase lag and the bandwidth of the desirable low frequency gain. As a rule of thumb, one may use  $a = 0.1 \omega_{c,d}$  as an initial guess, and iteratively improve from there. The worst phase lag for this choice of  $a$  is about  $5.7^\circ$ .

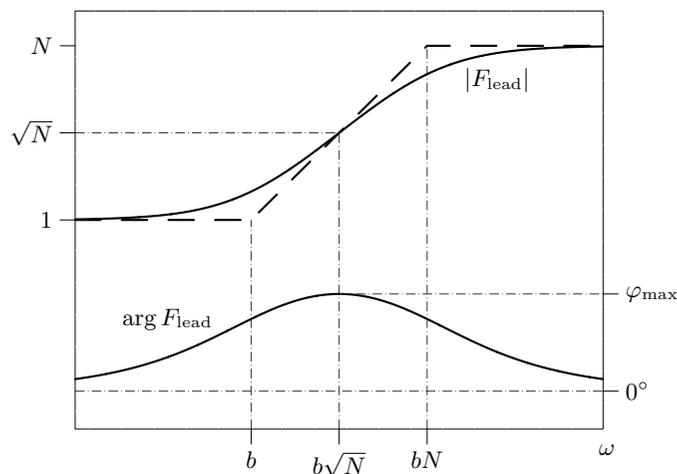


Figure 5.15a. Schematic Bode plot for the old parameterization of a lead compensator.

A read/write head of a hard disk is mounted on a mechanical arm which is moved by a motor. The system from motor input voltage to the angle of the arm is modelled by

$$Y(s) = \frac{5}{\tau_1 s + 1} \cdot \frac{0.05}{s(\tau_2 s + 1)} U(s)$$

where  $Y$  and  $U$  are the Laplace transforms of the output and input respectively. The numerical values of the constants are  $\tau_1 = 10^{-3}$  and  $\tau_2 = 0.05$ . The Bode plot of the system is given in Figure 5.15d.

- a) To begin with, assume that the arm is controlled using proportional feedback,

$$U(s) = K(R(s) - Y(s))$$

What are the step and ramp error coefficients (often referred to as  $e_0$  and  $e_1$ )? For what values of  $K$  are they defined?

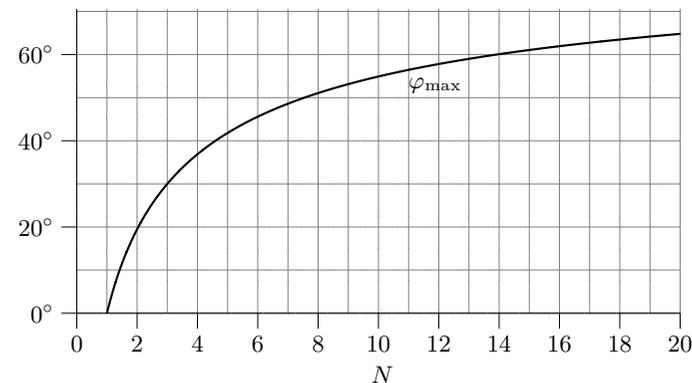


Figure 5.15b. Maximum phase advance as a function of  $N$  in a lead compensator parameterized the old way.

- b) Compute a controller,

$$U(s) = F(s)(R(s) - Y(s))$$

for the same system, such that the resulting system fulfills the following requirements:

- ◇  $e_0 = 0$
- ◇  $e_1 \leq 0.001$
- ◇  $\omega_c = 100$  rad/s
- ◇  $\varphi_m \geq 50^\circ$

5.16 A vehicle is described by the model

$$Y(s) = G(s)U(s)$$

where  $u$  is the input and  $y$  is the position and

$$G(s) = \frac{0.1}{s(s+1)^2}$$

The Bode diagram of the model is given in Figure 5.16a.

- a) Assume that the vehicle is controlled using proportional feedback

$$U(s) = K(R(s) - Y(s))$$

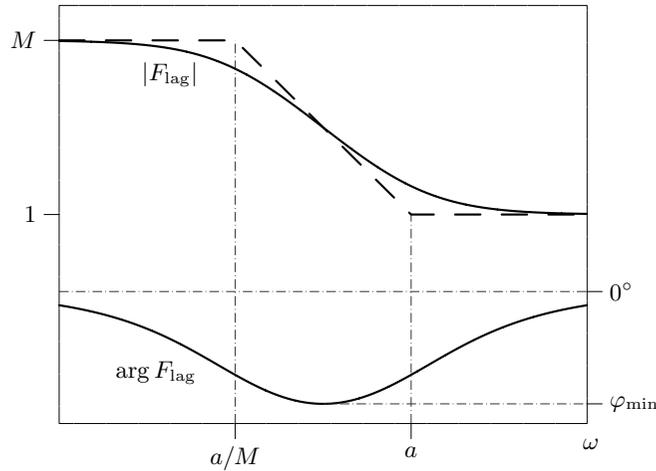


Figure 5.15c. Schematic Bode plot for the old parameterization of a lag compensator.

Which is highest cross-over frequency that can be achieved if it is required that the phase margin is at least  $60^\circ$ ? For which value of  $K$  is this cross-over frequency obtained?

- b) Assume that it is required that the vehicle is able to follow a reference path given by the function

$$r(t) = 0.5 \cdot t \quad t \geq 0$$

What is the resulting steady state error if the vehicle is controlled by the proportional feedback designed in a).

- c) Design a controller,

$$U(s) = F(s)(R(s) - Y(s))$$

for the vehicle above, such that the resulting control system fulfills the following requirements:

- ◊ The steady state error, using the same reference signal as in b), is less than 10% of what was achieved in problem b).
- ◊ The phase margin is at least  $60^\circ$ .
- ◊ The cross-over frequency is the same as what was obtained in problem a).

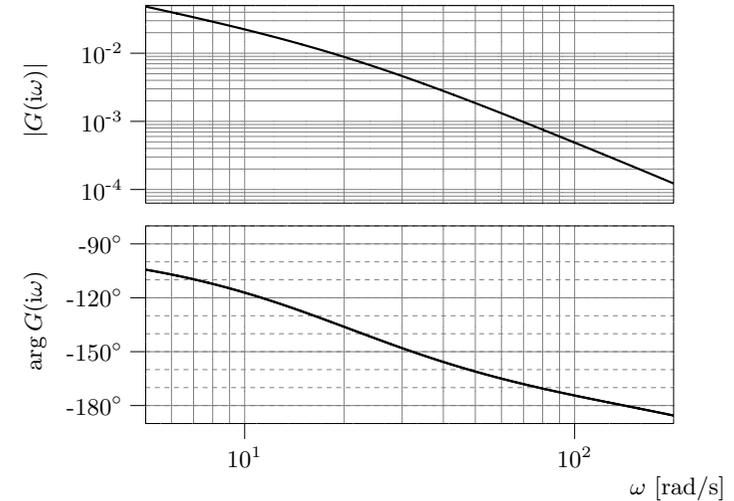


Figure 5.15d

- 5.17 The upper figure in Figure 5.17a shows, in four different cases, the Bode diagram for the loop gain

$$G_O(i\omega) = F(i\omega)G(i\omega)$$

and the lower figure in Figure 5.17a shows the amplitude curve,  $|G_C(i\omega)|$ , for the transfer function of the closed loop system

$$G_C(s) = \frac{G_O(s)}{1 + G_O(s)}$$

in the corresponding four cases. Combine the diagrams in Figure 5.17a.

- 5.18 A system is described by the relationship

$$Y(s) = G(s)U(s)$$

where the Bode diagram of  $G(s)$  is given in Figure 5.18a.

- a) Give a possible combination of values of  $p$ ,  $n$  and  $m$  such that the diagram corresponds to the transfer function

$$G(s) = \frac{K(s + z_1) \cdots (s + z_m)}{s^p (s + p_1) \cdots (s + p_n)}$$

b) Assume that the system is going to be controlled using the feedback

$$U(s) = F(s)(R(s) - Y(s))$$

Determine  $F(s)$  such that the control system fulfills the following requirements:

- ◇ The steady state error is zero when the reference signal is a unit step.
- ◇ The absolute value of the steady state error is less than 0.01 when the reference signal is a unit ramp.
- ◇ The compensated open loop system has cross-over frequency 3 rad/s and phase margin  $45^\circ$ .

5.19 A system is described by the model

$$Y(s) = \frac{2}{(10s + 1)^2} U(s)$$

and is controlled by the feedback

$$U(s) = K \frac{\tau_D s + 1}{\beta \tau_D s + 1} (R(s) - Y(s))$$

In Figure 5.19a, four step responses for the parameter pairs

- i)*  $K = 10 \quad \beta = 0.1$
- ii)*  $K = 10 \quad \beta = 0.8$
- iii)*  $K = 5 \quad \beta = 0.1$
- iv)*  $K = 5 \quad \beta = 0.8$

are shown. Combine these values with the step responses in Figure 5.19a.

5.20 Consider a system  $G(s)$ , whose Bode diagram of  $G(s)$  is given in Figure 5.20a, and the input-output relation  $Y(s) = G(s)U(s)$ . This system will be controlled with the feedback law

$$U(s) = F(s)(R(s) - Y(s)).$$

a) Calculate the closed-loop transfer function, i.e.,  $G_c(s)$  in the relation

$$Y(s) = G_c(s)R(s).$$

- b) Let  $F(s) = K$ . When  $K = 1$ , calculate crossover frequency  $\omega_c$  and phase margin  $\varphi_m$ . Is the closed-loop system stable for  $K = 1$ ?
- c) Using a controller of the form  $F(s) = K$ , calculate  $K$  such that the closed-loop system is twice as fast as with  $K = 1$ .
- d) What have we lost by making the closed-loop system twice as fast?
- e) Use a controller of the form

$$F_{\text{lead}}(s) = K \frac{\tau_D s + 1}{\tau_D \beta s + 1}$$

to obtain a closed-loop system that is twice as fast and has approximately the same overshoot as with  $K = 1$ .

- f) Calculate the steady-state error when the reference signal is a unit step. What must happen to  $F(0)$  if we want to decrease the steady-state error?
- g) If we use a controller of the form  $F(s) = F_{\text{lag}}(s)F_{\text{lead}}(s)$ , where

$$F_{\text{lag}}(s) = \frac{\tau_I s + 1}{\tau_I s + \gamma},$$

what is the value of  $\gamma$  such that the steady-state error is eliminated? What is the value of  $F(0)$  in that case?

- h) To avoid this, we decide that a steady-state error of 0.01 is acceptable when the input is a unit step. Choose  $\gamma$  to accomplish this, and an appropriate value of  $\tau_I$ . What can happen if we do not choose an appropriate value for  $\tau_I$ ?

### 5.21 2009-06-10 Upg. 3

Processer hämtade från kemisk industri kan ofta förenklat beskrivas av ett första ordningens system med tidsfördröjning. Ett exempel på ett sådant system är

$$G(s) = \frac{2}{s + 1} e^{-0.25s}$$

Man börjar med att reglera systemet med en P-regulator med  $K = 1/\sqrt{2}$ . Fasmarginalen för detta fall är acceptabel men systemet blir för långsamt.

För att upprätthålla produktkvaliteten kräver man att utsignalens värde stationärt inte avviker mer än 5% procent från det konstanta börvärdet.

Bestäm en regulator  $F(s)$  så att skärfrekvensen fördubblas jämfört med P-regleringen ovan, samt att de stationära kraven uppfylls. Systemet skall ha samma fasmarginal som vid den rena P-regleringen.

### 5.22 2011-06-10 Uppg. 4

Betrakta ett system med överföringsfunktion

$$G(s) = \frac{k_1}{(s+a)(s+b)(s+c)},$$

där  $k_1 = 100$ ,  $a = 3$ ,  $b = 6$  och  $c = 100$ . Bodediagrammet för  $G(s)$  visas i Figur 5.22a.

- Designa en fasavancerande och fasretarderande kompenseringslänk för  $G(s)$  så att skärfrekvensen blir 30 rad/s, fasmarginalen blir  $40^\circ$ , och statiska felet då referenssignalen är ett enhetssteg blir noll.
- Antag nu att det finns en tidsfördröjning i systemet så att öppna systemets verkliga överföringsfunktion ges av  $G^0(s) = G(s)e^{-T_d s}$ . För vilka värden på  $T_d$  är slutna systemet stabilt då regulatorn från Uppgift a) används?
- En regulator har nu designats för  $G(s)$  och förstärkningen för det resulterande slutna systemet  $G_c(s)$  återges i Figur 5.22b. För att reducera översvängen då steg i referenssignalen appliceras införs ett förfilter  $F_r(s) = \frac{1}{1 + \tau s}$  så att överföringsfunktionen från referens till utsignal blir  $G_c^0(s) = G_c(s)F_r(s)$ .

Uppskatta det minsta värdet av  $\tau$  som kommer att påverka systemets snabbhet vid steg i referenssignalen.

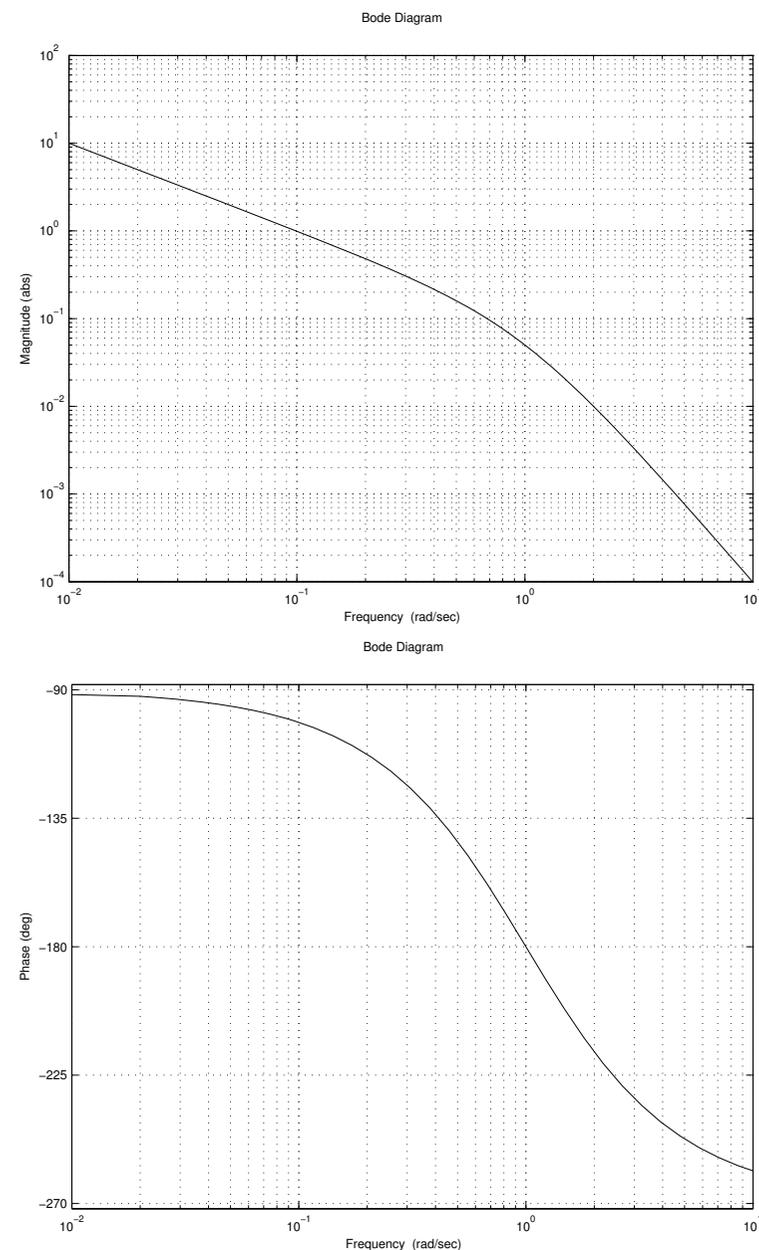


Figure 5.16a

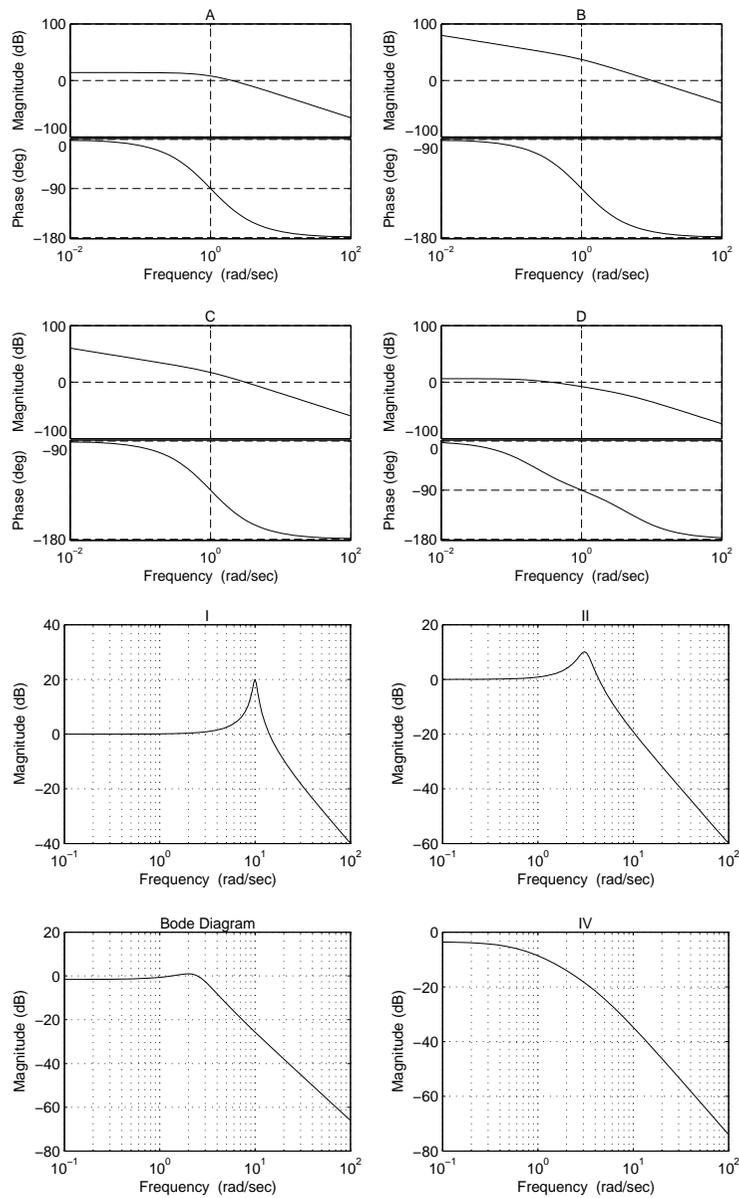


Figure 5.17a. Upper:  $G_O(i\omega)$ . Lower:  $|G_C(i\omega)|$ .

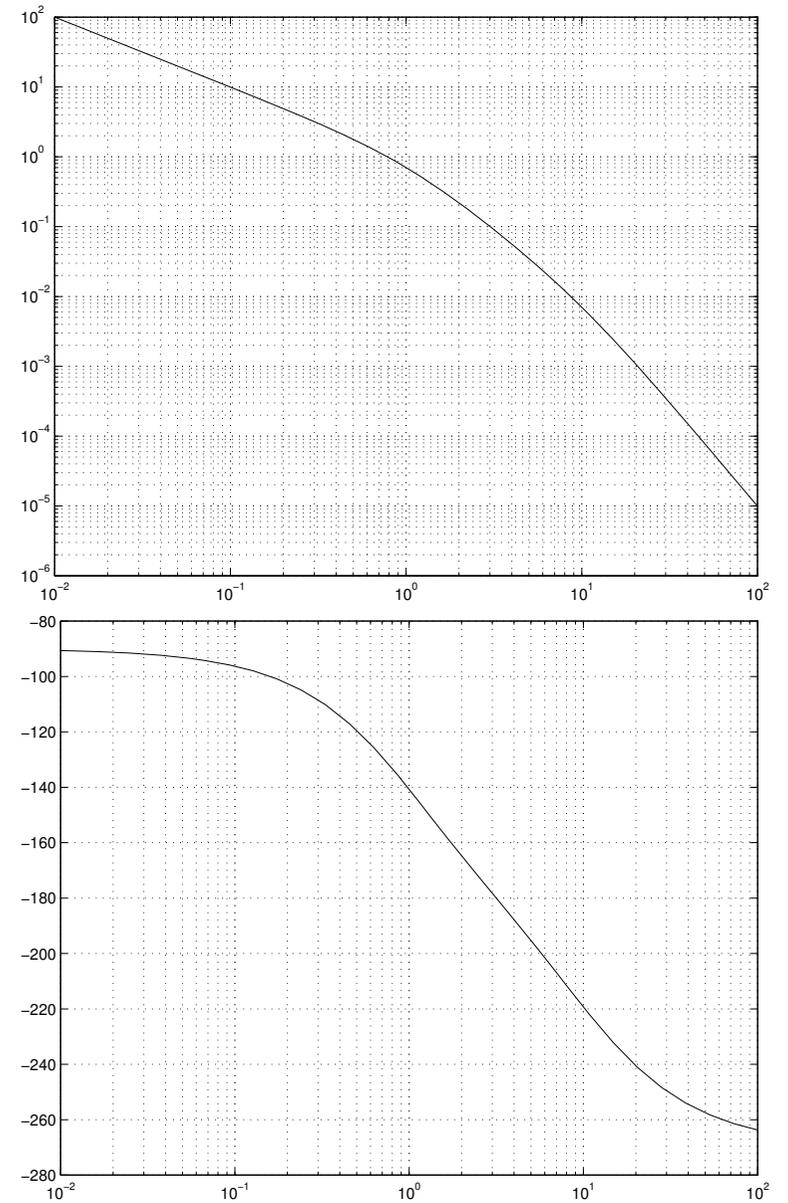


Figure 5.18a

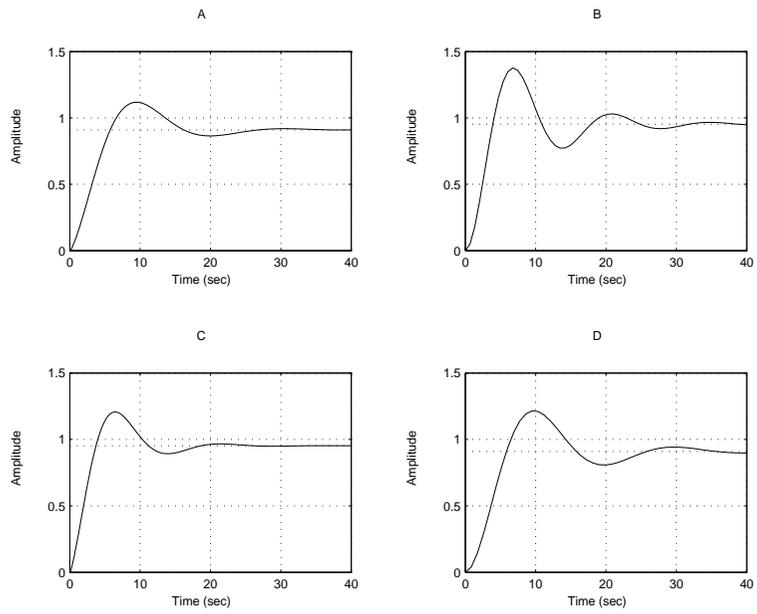


Figure 5.19a

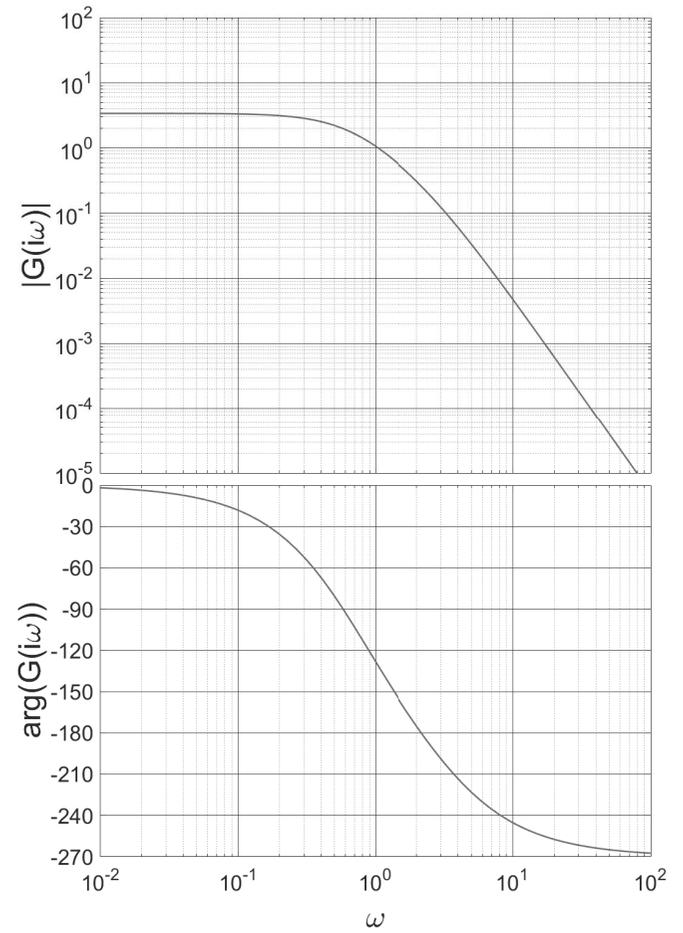


Figure 5.20a

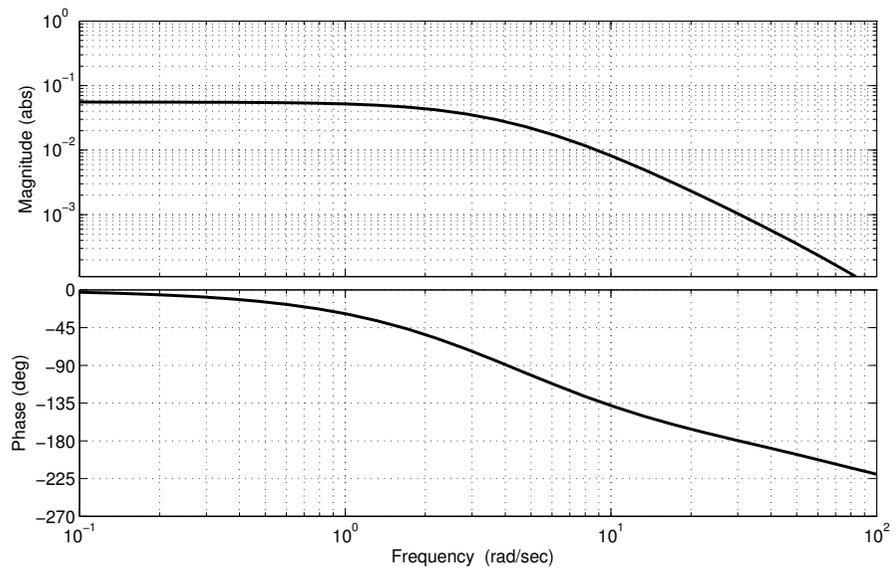


Figure 5.22a

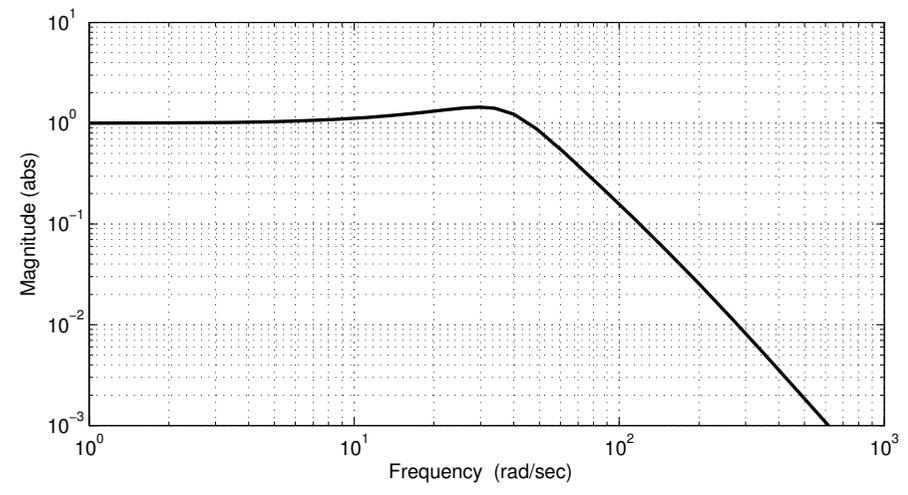


Figure 5.22b

## 6 Sensitivity and Robustness

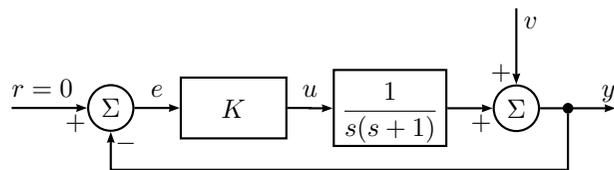


Figure 6.1a

- 6.1 Consider the control system in Figure 6.1a where  $v(t)$  is a sinusoidal disturbance,  $v(t) = \sin(t)$ . Compute the absolute value of the sensitivity function at  $\omega = 1$  rad/s as a function of  $K$ . How must  $K$  be selected if the amplitude of  $y(t)$  shall be less than the amplitude of  $v(t)$  at this frequency?

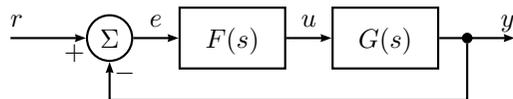


Figure 6.2a

- 6.2 Assume that we have constructed a controller  $F(s)$  for the model  $G(s)$ , see Figure 6.2a, such that there is no steady state error when the reference signal is a step. Let the real system be given by

$$G^0(s) = (s + 1)G(s)$$

and assume that  $G^0(s) \rightarrow 0$ ,  $s \rightarrow \infty$ . Also assume that the amplitude curve of the closed loop system has no resonance peaks and decreases, at least and asymptotically, with 20 dB<sub>20</sub>/decade for frequencies over the bandwidth. What is the highest possible bandwidth we can use for the closed loop system in Figure 6.2a, while at the same time guaranteeing stability?

- 6.3 Figure 6.3a shows a Nyquist diagram for the loop gain  $G_o$ . Show in a figure for what frequencies (that is, for what part of the Nyquist curve above) additive disturbances on the output are amplified in the sense that the output amplitude of the control system in Figure 6.3b is larger than the disturbance amplitude.

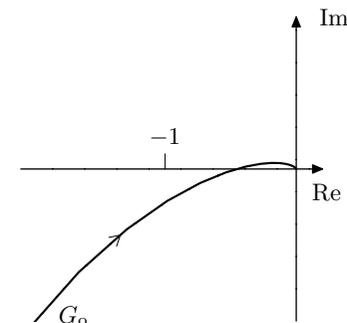


Figure 6.3a

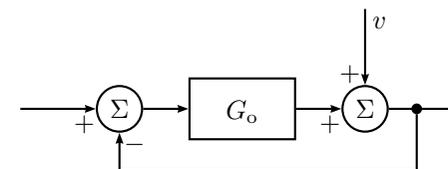


Figure 6.3b

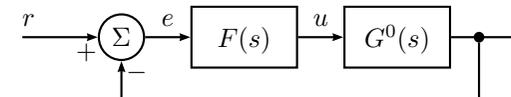


Figure 6.4a

- 6.4 Consider the control system in Figure 6.4a. The true system, denoted  $G^0(s)$ , is modeled as

$$G(s) = \frac{1}{s + 10}$$

The controller

$$F(s) = \frac{s + 10}{s}$$

gives an asymptotically stable closed loop system with the model  $G(s)$ . Now

assume that the system is given by

$$G^0(s) = G(s)(1 + G_\Delta(s))$$

where it is known that  $G_\Delta(s)$  has no poles in the right half plane, and that

$$|G_\Delta(i\omega)| < \frac{0.9}{\sqrt{1 + \omega^2}}$$

Can we be sure that the closed loop system is asymptotically stable?

6.5 A process is described by the model  $G(s)$ , while the process in reality has the transfer function

$$G^0(s) = e^{-sT}G(s)$$

a) Draw the absolute value of the inverse of the relative model error, that is,

$$\frac{1}{|G_\Delta(i\omega)|}$$

b) Assume that we design a controller  $F(s)$  starting with the model  $G(s)$ . How large may

$$\left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right|$$

be at most, in order to guarantee asymptotic stability of the closed loop system for all values of  $T$ , when the controller  $F(s)$  is used on the system  $G^0(s)$ ?

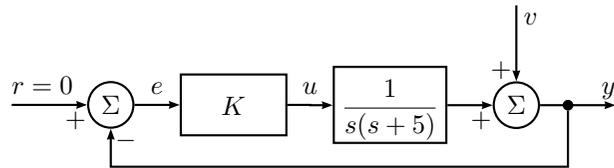


Figure 6.6a

6.6 Consider the control system in Figure 6.6a.

a) Assume that the real system is given by

$$G^0(s) = G(s) + \tilde{G}(s),$$

where

$$G(s) = \frac{1}{s(s+5)}$$

and let  $K = 25/2$ . Use the robustness criterion to obtain a condition on  $|\tilde{G}(i\omega)|$  that guarantees stability in the closed loop system. Does  $\tilde{G}(s) = 1$  fulfill the conditions?

b) Now let  $\tilde{G}(s) = \alpha$  where  $\alpha$  is a scalar. Calculate the characteristic equation for the closed loop system and decide for which  $\alpha$  the system is stable. Does this contradict the condition from the robustness criterion?

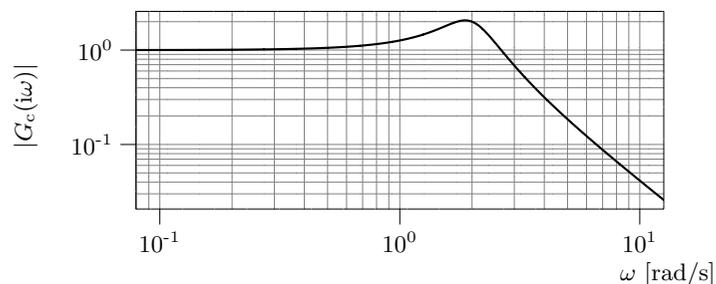


Figure 6.7a

6.7 A DC-motor is assumed to have the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

and it is controlled using proportional feedback,

$$U(s) = F(s)(R(s) - Y(s))$$

where  $F(s) = 4$ . The amplitude curve of the feedback system

$$|G_c(i\omega)| = \left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right|$$

is given in Figure 6.7a. Assume that the real system is given by

$$G^0(s) = G(s) \frac{\alpha}{s + \alpha}, \quad \alpha > 0$$

and the controller  $F(s)$  is used on the system  $G^0(s)$ .

- Draw a root locus with respect to  $\alpha$  for the characteristic equation of the closed loop system and determine for which  $\alpha$  the closed loop system is asymptotically stable.
- Use the robustness criterion to decide for which  $\alpha$  the closed loop system is asymptotically stable.
- Comment on the possible differences in the demands on  $\alpha$  in a) and b).

6.8 A system  $G^0(s)$  is controlled using a regulator  $F(s)$ . In Figure 6.8a the amplitude part the Bode plot of the nominal closed loop system,

$$G_c(s) = \frac{F(s)G(s)}{1 + F(s)G(s)}$$

is shown. It is known that  $G_c$  is stable, and it is assumed that  $G$  and  $G^0$  have the same number of poles in the right half plane. The model uncertainty  $G_\Delta(s)$ , defined by

$$G_\Delta = \frac{G^0 - G}{G}$$

is assumed bounded by  $|G_\Delta(i\omega)| \leq \gamma\omega$ . In what interval must  $\gamma$  lie to guarantee stability of the closed loop system?

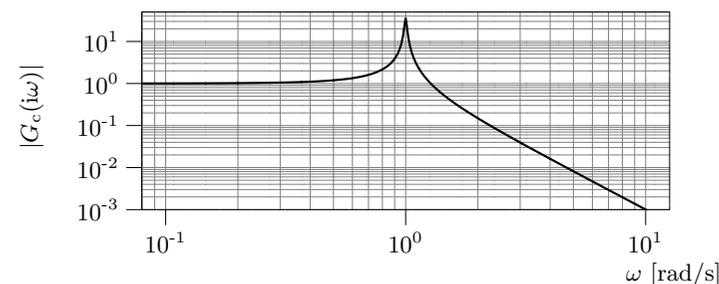


Figure 6.8a

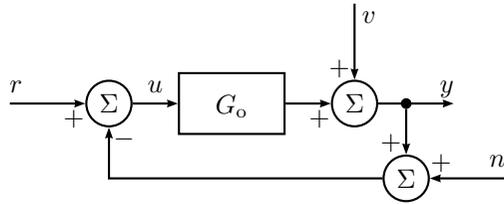


Figure 6.9a

6.9 Consider the system in Figure 6.9a. For  $r(t) = 0$ ,  $n(t) = 0$  and  $v(t) = \sin t$  the steady-state output is given by

$$y(t) = \frac{1}{\sqrt{2}} \sin\left(t - \frac{\pi}{4}\right)$$

Determine the steady-state output  $y(t)$  when  $r(t) = 0$ ,  $v(t) = 0$  and  $n(t) = \sin t$ .

6.10 Recall the model that was used in the design of a lead-lag controller using MATLAB in Problem 5.13. Assume that the true system contains a time constant that was neglected, and that the transfer function of the system is given by

$$G^0(s) = G(s) \frac{1}{s+1}$$

- Determine the relative model error  $G_\Delta(s)$ .
- Draw  $\frac{1}{|G_\Delta(i\omega)|}$  and  $\left| \frac{F(i\omega)G(i\omega)}{1+F(i\omega)G(i\omega)} \right|$  in a Bode plot, when  $G(s)$  is given by

$$G(s) = \frac{725}{(s+1)(s+2.5)(s+25)}$$

for the two cases  $F(s) = 1$  and  $F(s)$  being the controller designed in Problem 5.13. What can be said about the robustness of the closed loop system in these two cases when  $F(s)$  is used for control of the “true” system  $G^0(s)$ ?

One possible solution to the design problem in Problem 5.13 was the controller

$$F(s) = 0.46 \cdot \frac{0.43s+1}{0.090s+1} \cdot \frac{2.0s+1}{2.0s}$$

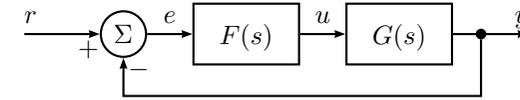


Figure 6.11a

6.11 Consider the control system in Figure 6.11a. The controller  $F(s) = 2$  gives the Bode plot of the loop gain  $F(i\omega)G(i\omega)$  shown in Figure 6.11b. The Bode plot of the sensitivity function is shown in Figure 6.11c. The reference signal is  $r(t) = 2 \sin 0.1t$ . Determine the amplitude of the error in steady state.

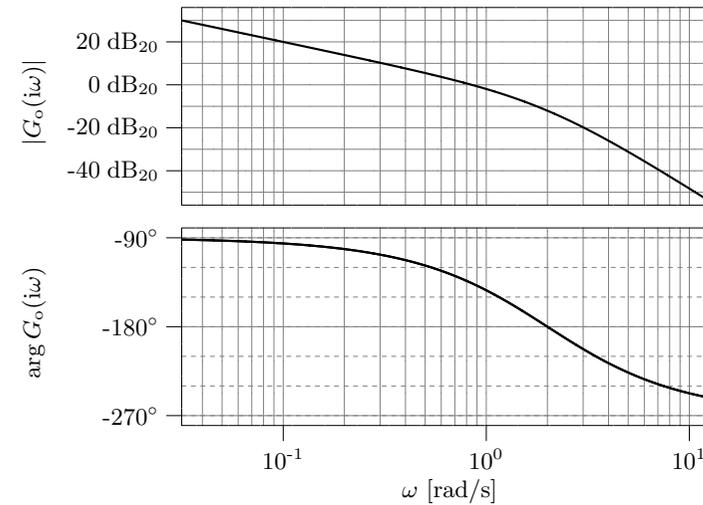


Figure 6.11b

6.12 In Problem 5.14 we saw how the amount of dissolved oxygen depends on the stirring speed. A lead-lag controller was designed for the model linearized around 1200 r/min. Check if this controller also stabilizes the system when the stirring speed is 400 r/min.

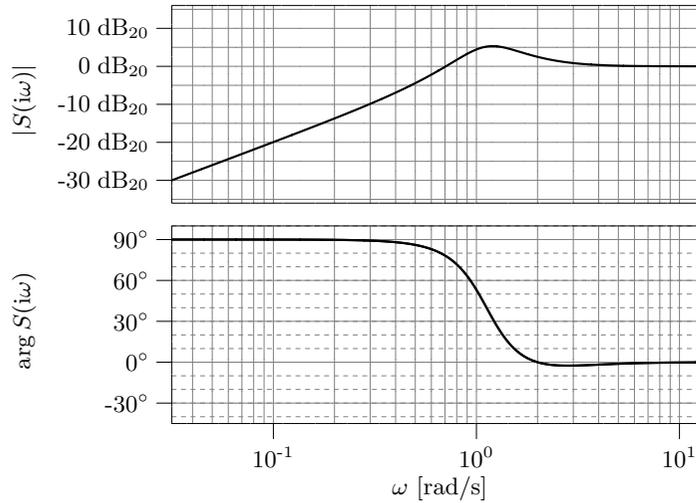


Figure 6.11c

6.13 Consider the connected tank system in Figure 6.13a where  $u$  is the inflow to the upper tank and  $y$  is the level in the lower tank. The system can approximately be described by the following transfer function

$$G(s) = \frac{1}{(s+1)^2}$$

The level in the lower tank is controlled by a P controller;

$$U(s) = K(R(s) - Y(s))$$

The goal of the control is to minimize the influence of the disturbance  $v$ . This has been formalized as the following demand on the system:

$$|S(i\omega)| \leq 0.1$$

at  $\omega = 1$ , where  $S(s)$  denotes the sensitivity function. How must  $K$  be selected if the demand should be satisfied?

6.14 **2010-12-14 Upg. 5ab**

För det återkopplade systemet i Figur 6.14a definieras känslighetsfunktionen

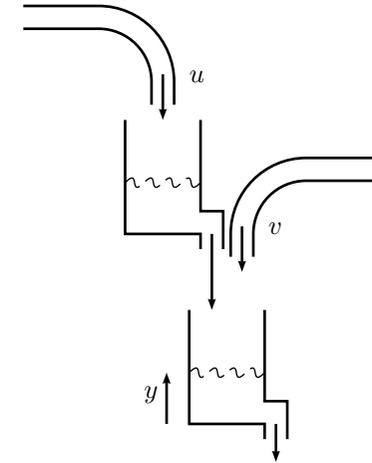


Figure 6.13a

$S(s)$  och slutna systemets överföringsfunktion  $G_c(s)$  som

$$S(s) = \frac{1}{1 + G_o(s)} \quad \text{och} \quad G_c(s) = \frac{G_o(s)}{1 + G_o(s)},$$

där  $G_o(s)$  är det öppna systemets överföringsfunktion.

- Förklara hur  $|S(i\omega)|$  påverkar ett återkopplat systems förmåga att undertrycka en additiv störning  $v$  till utsignalen, samt hur  $|G_c(i\omega)|$  påverkar ett återkopplat systems förmåga att undertrycka mätbrus  $n$ . Kan både störningen och mätbruset undertryckas godtyckligt mycket samtidigt? Motivera ditt svar.
- Visa att om  $r$  är ett steg så blir statistiska reglerfelet noll ifall  $S(s)$  har alla poler strikt i vänstra komplexa halvplanet och ett nollställe i origo.

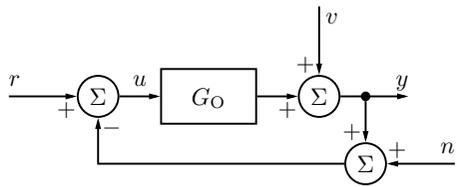


Figure 6.14a

# 7 Special Controller Structures

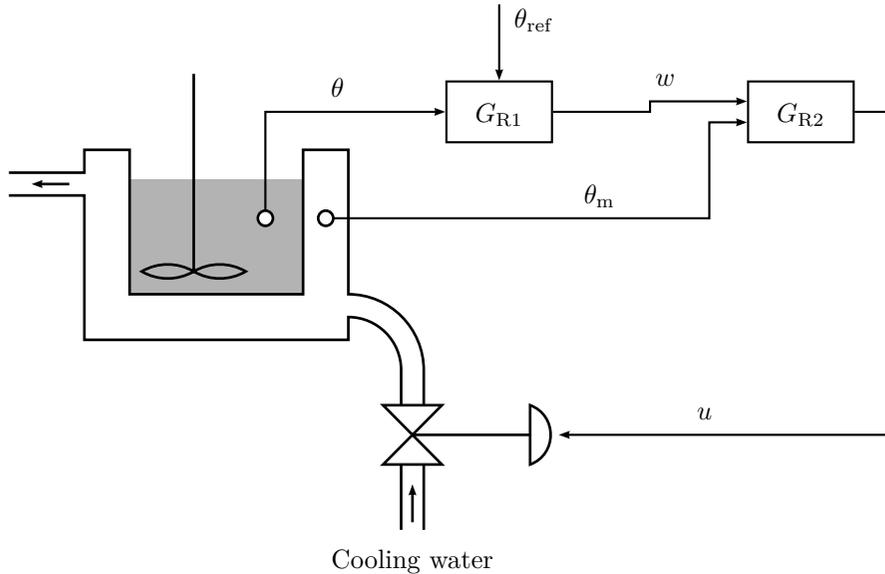


Figure 7.1a

7.1 To control the temperature  $\theta$  in a chemical reactor, the control system in Figure 7.1a is used, where  $\theta_{\text{ref}}$  is the desired (reference) temperature. The temperatures  $\theta$  and  $\theta_m$  in the reactor and the cooler, respectively, are measurable and can be used to control the valve  $u$ . The structure of the control system is given by Figure 7.1b. It is here assumed that both  $G_{R1}$  and  $G_{R2}$  are P controllers.

- a) Let  $K_2 = 9$  and draw the Bode plot of the transfer function from  $w$  to  $\theta$ . Then choose  $K_1$  so that the gain margin  $A_m = 2$ . What are the gain crossover frequency  $\omega_c$  and the steady-state error, if we assume that  $\theta_{\text{ref}}$  is changed stepwise?

- b) Suppose that we make a simple feedback loop instead, see Figure 7.1c. How is the Bode plot affected? Again, choose  $K_1$  so that the gain margin  $A_m = 2$ , and determine the gain crossover frequency  $\omega_c$  and the steady-state error. Compare with a) with respect to steady-state errors and response times. Conclusions?

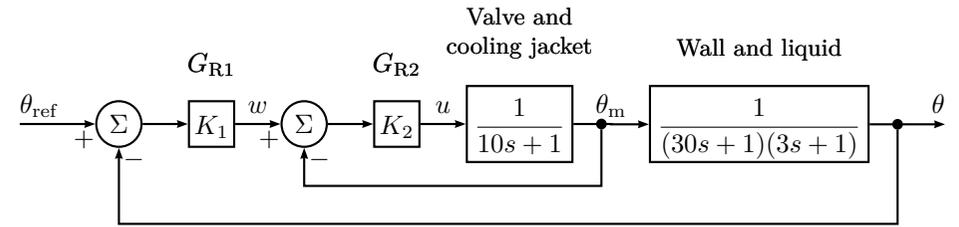


Figure 7.1b

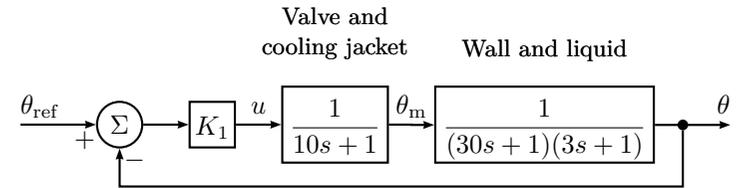


Figure 7.1c

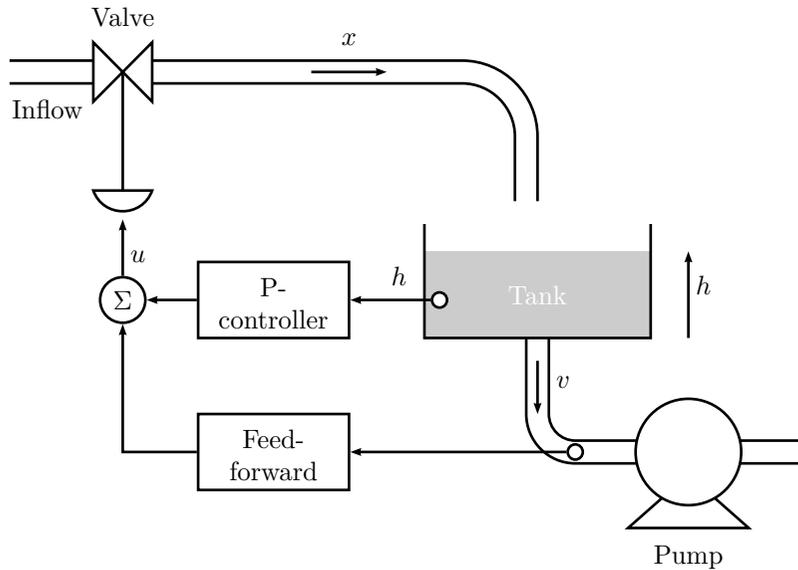


Figure 7.2a

7.2 A level control system for a water tank is shown in Figure 7.2a, where all variables denote offsets from an operation point. The inflow,  $x(t)$ , to the tank (with area  $A$ ) is determined by the valve, and the outflow,  $v(t)$ , is determined by the pump. Stu Dent has got the assignment to keep the water level in the tank constant, in spite of variations in the outflow  $v(t)$ . First, Stu determines the transfer function  $G_v(s)$  from the valve to  $x(t)$ . By step response experiments, he obtains the following result:

$$G_v(s) = \frac{1}{0.5s + 1}$$

- a) Because the disturbance  $v(t)$  is measurable, Stu first considers a feedforward compensator to completely eliminate it. Stu, who also knows that it is dangerous to differentiate the disturbance, cancels all the derivative terms in the compensator. Compute the feedforward compensator, and determine the response  $h(t)$  Stu will get, if the outflow  $v(t)$  is changed stepwise with an amplitude of 0.1.

- b) To improve the control system, Stu also introduces a proportional feedback of the water level  $h$ . What is the steady-state error in the level  $h$  now, if the outflow is changed in the same way as in a)?

7.3 Consider the following system

$$Y(s) = \frac{2}{s+3}U(s) + \frac{3}{s+4}V(s)$$

where  $u$  is the control signal,  $y$  is the output and  $v$  is a disturbance. It is desired that  $y$  should be as small as possible despite the disturbance  $v$ .

- a) Design a feedforward controller from  $v$  to  $u$  that eliminates the influence of  $v$  on  $y$ .
- b) Assume that  $v$  is a pure sinusoid with amplitude 2. How large will the control signal be?
- c) The real system is described by

$$Y(s) = \frac{b}{s+3}U(s) + \frac{3}{s+4}V(s)$$

where  $b$  value is not exactly known but has its value close to 2. To solve this problem a P controller is added to the feedforward controller that was designed in a). The full controller looks like

$$U(s) = -KY(s) + F_f(s)V(s)$$

where  $F_f(s)$  is the feedforward controller. What is the stationary error if  $v = 1$ ?

7.4 The transfer function for a temperature control system is given by

$$Y(s) = \frac{3}{s+1}U(s) + \frac{4}{(s+2)(s+5)}V(s)$$

where  $y$  is the controlled temperature,  $u$  is the supplied power and  $v$  is the temperature of the surroundings. Assume that the desired temperature is zero.

- Design a feedforward controller  $U(s) = F_f(s)V(s)$  which eliminates the influence of the disturbance  $v$  on  $y$ .
- To simplify implementation  $F_f(s)$  is replaced with a constant,  $\tilde{F}_f = F_f(0)$ . Assume that  $v$  is given by  $v(t) = -1 - 0.1t$  and that  $U(s) = \tilde{F}_f V(s)$  is used. What will  $y(t)$  be in steady state?
- The previous controller is now extended with a P controller:

$$U(s) = \tilde{F}_f V(s) - KY(s)$$

What will now  $y(t)$  be in steady state?

- Assume that one only uses the P controller

$$U(s) = -KY(s)$$

What will now  $y(t)$  be in steady state?

#### 7.5 2011-06-10 Upg. 1ab

- I Figur 7.5a visas en process och en regulator med både framkopplings- och återkopplings-länkar. Härled (uttryck i  $G_1(s)$ ,  $G_2(s)$ ,  $F_y(s)$ ,  $F_r(s)$  och  $F_f(s)$ )
  - överföringsfunktionen från referenssignalen  $r$  till utsignalen  $y$ ,
  - överföringsfunktionen från störsignalen  $d$  till utsignalen  $y$ .
- Antag att processen i Figur 7.5a har överföringsfunktionerna  $G_1(s) = \frac{s+2}{s^2+2s+1}$  och  $G_2(s) = \frac{1}{s+3}$ . Ta fram en lämplig framkoppling  $F_f(s)$  för att eliminera inverkan av störningen  $d$ . Diskutera även hur du enkelt kan implementera framkopplingen om målet är att eliminera konstanta störsignaler i stationäritet.

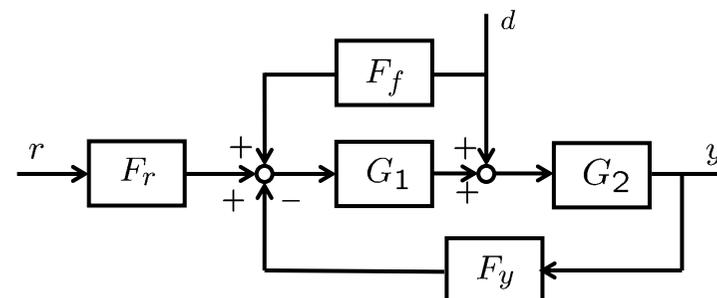


Figure 7.5a

# 8 State Space Description

8.1 Define suitable state space variables for the DC motor discussed in Problem 2.1, and write the system in state space form.

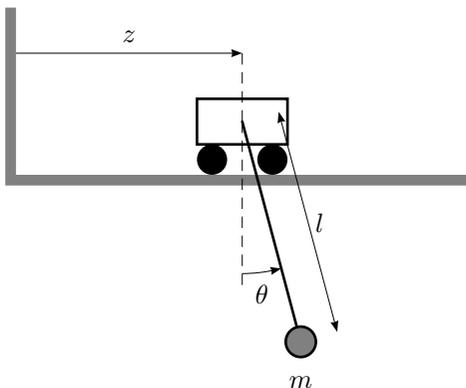


Figure 8.2a

8.2 Consider the system illustrated in Figure 8.2a. It consists of a hinge that can move in the direction marked “z”, and a thereto attached pendulum. The system is described by the equation

$$l\ddot{\theta} + g \sin \theta + \ddot{z} \cos \theta = 0$$

Define state space variables, input, and output as

$$x_1 = \theta \quad x_2 = \dot{\theta} \quad u = \ddot{z}/\ell \quad y = \theta$$

and

$$\omega_0^2 = g/\ell$$

Linearize the system around the equilibrium point given by

$$x_1 = \pi \quad x_2 = 0 \quad u = 0$$

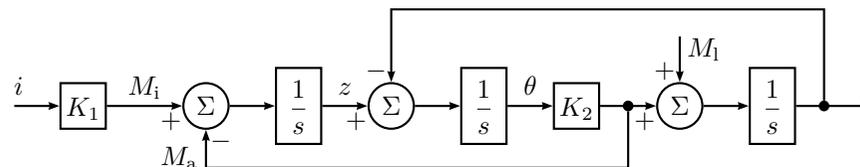


Figure 8.3a

8.3 The block diagram in Figure 8.3a describes an electric motor that drives a load via an elastic axis. Here  $i$  is the driving current to the motor, which gives the torque  $M_i$ .  $z$  is the turning rate of the motor and  $y$  is the turning rate of the load.  $\theta$  is the angle of the transmission axis.  $M_a = K_2\theta$  is the torque this angle causes.  $M_1$  is the torque from the load. Give a state space description for the system with  $M_1$  and  $i$  as inputs and  $y$  as output. (There are at least two different ways to solve this problem.)

8.4 Write the following systems in state space form.

a)

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 6u$$

b)

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y = 4\frac{d^2u}{dt^2} + \frac{du}{dt} + 2u$$

c)

$$G(s) = \frac{2s + 3}{s^2 + 5s + 6}$$

Use for example controllable or observable canonical form or diagonal form.

8.5 A system has the impulse response (weight function)

$$g(t) = 2e^{-t} + 3e^{-4t}$$

Write the system in state space form.

8.6 Consider the system

$$\dot{x} = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

$$y = (-1 \quad 2) x$$

Compute the transfer function of the system.

8.7 Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

The input is being held constant,  $u = u_0$ , for the time  $t_0 \leq t \leq t_0 + T$ .

Give a relation between  $x(t_0)$ ,  $x(t_0 + T)$ ,  $y(t_0)$ ,  $y(t_0 + T)$  and  $u_0$ .

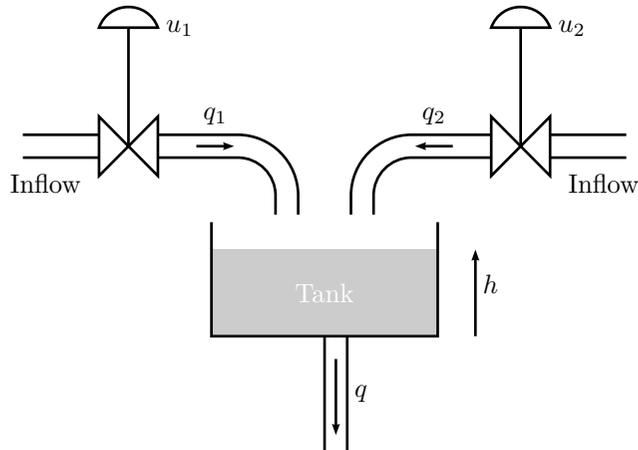


Figure 8.8a

8.8 Consider the tank in Figure 8.8a. The tank can be filled from two different pipes, where the flows  $q_1$  and  $q_2$  are determined by the valve settings  $u_1$  and  $u_2$ . If  $q_1$ ,  $q_2$ ,  $u_1$ ,  $u_2$ , and  $h$  denote the deviation from a nominal value, we get the linearized equation

$$\dot{h} + \frac{1}{\tau} h = u_1 + u_2$$

where  $\tau = 1$ . It is desired that the level should follow a reference value  $h_{\text{ref}}$  and that  $q_1$  and  $q_2$  should be of approximately the same size. Therefore two PI controllers are used so that

$$u_1 = (h_{\text{ref}} - h) + \int_0^t (h_{\text{ref}} - h) d\tau \quad (8.1)$$

$$u_2 = (h_{\text{ref}} - h) + \int_0^t (h_{\text{ref}} - h) d\tau \quad (8.2)$$

- Introduce the state variable  $x_1 = h$ , and let  $x_2$  and  $x_3$  represent the integrals in (8.1) and (8.2) respectively. Derive a state space description of the closed loop system with  $h_{\text{ref}}$  as input and  $h$  as output.
- Verify that the closed loop system is unobservable and that the unobservable subspace is spanned by the vector

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Give a practical interpretation of this phenomenon.

- The level is measured by two different sensors, and due to the poor accuracy in the first sensor it delivers the level signal together with an error. The equation of the first regulator can hence be written

$$u_1 = -(h + n) + \int_0^t -(h + n) d\tau \quad (8.3)$$

where it for simplicity has been assumed that  $h_{\text{ref}} = 0$ . The second regulator is then given by

$$u_2 = -h + \int_0^t -h d\tau \quad (8.4)$$

Modify the state space model by letting the measurement disturbance be the input to the state space model of the closed loop system.

8.9 Consider the system

$$\dot{x} = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

Is it possible to control the system from the origin to  $x = (1 \ 3)^T$  within 4 seconds?

8.10 Give the dimensions of the controllable and unobservable subspaces to the systems below. Give also the controllable and unobservable subspaces.

a)

$$\dot{x} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} u$$

$$y = (1 \ 3 \ 1.5) x$$

b)

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix} x + \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} u$$

$$y = (0 \ 3 \ 0) x$$

8.11 A state space representation of

$$G(s) = \frac{1}{s+1}$$

is given by

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

$$y = (1 \ 0) x$$

a) Compute  $x_1(t)$ ,  $x_2(t)$  and  $y(t)$  if  $x(0) = 0$  and

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

b) Is the system asymptotically stable? Input-output-stable?

c) Examine the controllability and observability for the system.

d) Explain why the realization is not suitable for simulating a system whose transfer function is  $G(s)$ .

8.12 Compute the poles and zeros of the system

$$\dot{x} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$$y = (1 \ 1) x$$

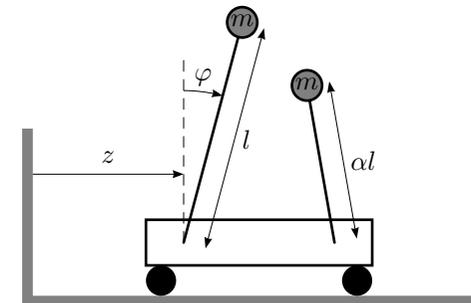


Figure 8.13a

8.13 Two mathematical pendulums are mounted on a trolley. They are mounted so that they can move without friction in a plane coinciding with the direction of movement for the trolley. The lengths of the pendulums are  $l$  and  $\alpha l$  and their masses are  $m$ . For one pendulum we have

$$\ddot{z} \cos \varphi + \ddot{\varphi} l = g \sin \varphi$$

a) Linearize the system around  $\varphi = 0$  and put the constants  $l$ ,  $m$ , and  $g$  to 1 and write the equations in the form  $\dot{x} = Ax + Bu$ .

b) Give the values on  $\alpha$  for which the system is controllable. Give a practical motivation.

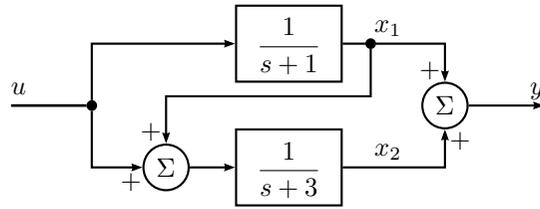


Figure 8.14a

8.14 A system is given by the block diagram in Figure 8.14a. Derive a state space model of the system, with the state space variables given in the figure.

8.15 The substances A and B react according to,  $3A \rightarrow B$ , in a tank. The reaction speed is given by  $r_A = -k_1 c_A^3$ . The inflow,  $q$  to the tank has concentration  $c_{A,in}$ . The tank volume  $V$  and the in- and outflow can be considered constant.

- Determine the dynamical mass balance for the components A and B in the form of differential equations.
- Linearize the differential equations around a stationary point,  $c_A^*$ ,  $c_B^*$ ,  $c_{A,in}^*$ , and use the state space representation

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx + Du$$

where the state  $x$  consists of the deviations  $c_{A,\Delta}$  and  $c_{B,\Delta}$  of the concentrations. The input signal  $u$  is the deviation  $c_{A,in,\Delta}$  in the inflow concentration and the output signal  $y$  is the deviation  $B_{c_{B,\Delta}}$  in concentration of component.

8.16 Figure 8.16a shows a protein that changes between the conformations  $x_1$  and  $x_2$ . The kinetic parameters  $k_{ij}$  determine the rate of the conformational changes. The input signal,  $u$ , together with the constant,  $K$ , represent introduction of  $x_2$  to the system. We assume that we can measure  $x_2$ , i.e.  $y = x_2$ . All reaction rates are assumed to be given by expressions proportional to the concentration of the states they emanate from. That is, for a state  $i$  the differential equation will be  $\dot{x}_i = \sum_p k_{ip} x_p - \sum_q k_{qi} x_i$ , where  $p$  is the number of incoming and  $q$  the number of outgoing flows from the state  $i$ . This corresponds to mass action kinetics for a regular biochemical reaction network.

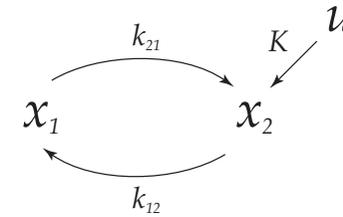


Figure 8.16a. Protein in two possible conformations.

- Write down the state space equations that describe the system.
- Compute the poles of the system. Is the system stable?
- What are the conditions on the kinetic parameters  $k_{ij}$  for the system to be observable? Give an interpretation of this result.
- Assume that the conformation change from  $x_2$  to  $x_1$  is catalyzed by an enzyme. To describe this, include Michaelis-Menten saturation in the flow from  $x_2$  to  $x_1$ , i.e. replace the expression  $k_{12}x_2$  with  $\frac{V_{max}x_2}{K_M+x_2}$ , where  $V_{max}$  is the maximal rate and  $K_M$  the saturation constant. Write down the differential equations for the states  $x_1$  and  $x_2$ .
- Sketch the relation between  $x_2$  and the saturated reaction rate in d).
- Linearize the system around the stationary point  $x_1 = \frac{V_{max}}{2k_{21}}$ ,  $x_2 = K_M$ ,  $u = 0$ .

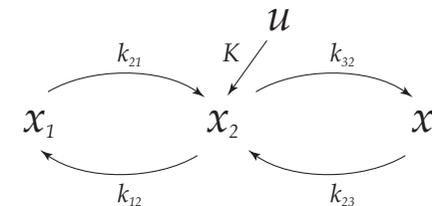


Figure 8.16b. Protein in three possible conformations.

Now, assume that the protein can exist in a third conformation  $x_3$ , according to Figure 8.16b. We measure  $y = x_2$  as before.

- Write down the state space equations that describe the system.

- h) What are the conditions on the kinetic parameters  $k_{ij}$  for the system to be observable? Give an interpretation of this result.

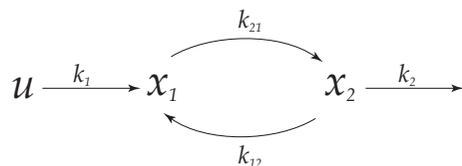


Figure 8.17a

- 8.17 The model in Figure 8.17a describes the interaction between two proteins with concentrations  $x_1$  and  $x_2$ , where the kinetic parameters  $k_1, k_{21}, k_{12}$  and  $k_2$  determine the reaction rate.

The law of mass actions for biochemical reaction networks leads to the following state space model for the system.

$$\dot{x} = \begin{pmatrix} -k_{21} & k_{12} \\ k_{21} & -k_{12} - k_2 \end{pmatrix} x + \begin{pmatrix} k_1 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x.$$

The input  $u$  is a unit step.

- What are the steady state values of  $x_1$  and  $x_2$ , when the input  $u$  is a unit step?
  - Compute the transfer function of the system, from  $u$  to  $y$ .
  - Use the final value theorem to find the steady state gain of  $y$ . Compare with the result in a). What conclusions can be drawn?
- 8.18 An electromagnet can make a metal ball levitate. The electromagnet is positioned so as to make the magnetic force act upwards in the vertical plane. By placing a metal ball under the magnet, the ball stays in the air if the magnetic force exactly matches the gravity acting on the ball.

The gravitational force is  $F_g = mg$  where  $m$  is the mass of the ball and  $g$  is the gravitational constant. The force from the magnet is described by  $F_m(t) = ku(t)y^{-2}(t)$  where  $u(t)$  is the current through the coil of the electromagnet,

$y(t)$  is the distance from the magnet to the ball and  $k$  is a proportionality constant. Force equilibrium leads to  $m\ddot{y}(t) = -\frac{ku(t)}{y^2(t)} + mg$ .

Using the states  $x_1 = y$  and  $x_2 = \dot{y}$  leads to the state space description

$$\dot{x}(t) = \begin{pmatrix} x_2(t) \\ -\frac{ku(t)}{mx_1^2(t)} + g \end{pmatrix}$$

$$y(t) = x_1(t)$$

- Show that all stationary points are given by  $x_1 = x_{10}, x_2 = 0$  and  $u_0 = g \frac{mx_{10}^2}{k}$ .
- Determine a linear state space model approximating the nonlinear system around each stationary point.

### 8.19 2008-12-20 Uppg. 5

Studera det olinjära systemet

$$\dot{y}(t) = -y(t)u(t) + v$$

med insignal  $u(t)$ , utsignal  $y(t)$  och där  $v$  är en okänd konstant positiv störning ( $v > 0$ ). Målet är att konstruera en regulator, som håller utsignalen  $y(t)$  på en given konstant nivå  $y(t) = y_0, y_0 > 0$ .

- Linjärisera systemet runt motsvarande stationära punkt.
- Uppgiften är att konstruera en regulator så att det återkopplade linjäriserade systemet med styrlag

$$U(s) = F(s)E(s), \quad F(s) = K \frac{\tau_I s + 1}{\tau_I s}, \quad e(t) = y_0 - y(t)$$

är stabilt för alla positiva värden på den okända störningen  $v$ .

Visa att det är möjligt samt ange ett val av  $-\infty < K < \infty$  och  $\tau_I > 0$ .

### 8.20 2011-12-17 Uppg. 4abc

Betrakta systemet

$$\dot{x}_1(t) = x_1(t) + u(t)$$

$$\dot{x}_2(t) = -2x_1(t) + \alpha x_2(t) - u(t)$$

$$y(t) = x_2(t).$$

- a) För vilka värden på  $\alpha$  är det öppna systemet (d.v.s. då  $u(t) = 0$  för alla  $t$ ) asymptotiskt stabilt?
- b) För vilka värden på  $\alpha$  är systemet observerbart?
- c) För vilka värden på  $\alpha$  är systemet styrbart?

# 9 State Feedback

9.1 Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= (1 \quad 0) x \end{aligned}$$

- Calculate a state feedback that places the poles in I)  $\{-3, -5\}$ , II)  $\{-10, -15\}$ . What limits the possibility to achieve arbitrary dynamics of the closed loop system?
- Suppose only the output is measured. Calculate an observer that makes the transfer function from the reference signal to the output the same as in a). Discuss the influence of the poles of the observer.

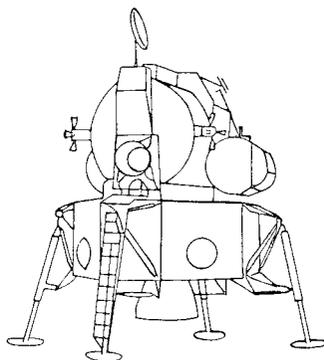


Figure 9.2a

9.2 Figure 9.2a shows the Lunar Excursion Module from the Apollo project. Consider the module hovering a short distance above the surface of the moon using its main engine. If the pitch angle of the module (angle between the vertical line and the direction of movement) differs from zero, a horizontal component of the force is obtained and the module is accelerating along the surface.

We will study a block diagram which shows the connection between the input  $u$  (the control signal to the attitude thrusters), the pitch angle  $\theta$  and the position coordinate  $z$ . See Figures 9.2b and 9.2c.

The module is both in the  $\theta$ -direction and in the  $z$ -direction obeying Newton's law of motion without any kind of damping. The transfer function from the control signal of the astronaut ( $y_{\text{ref}}$ ) to velocity  $\dot{z}$  is

$$\frac{K_1 K_2}{s^3}$$

which is very difficult to control by hand.

- Write the system in state space form.

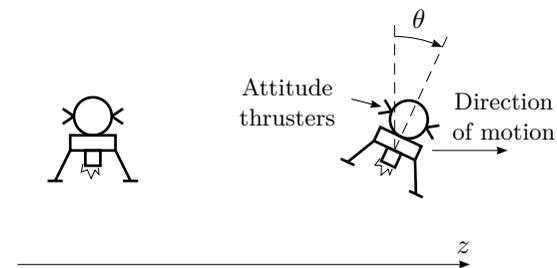


Figure 9.2b

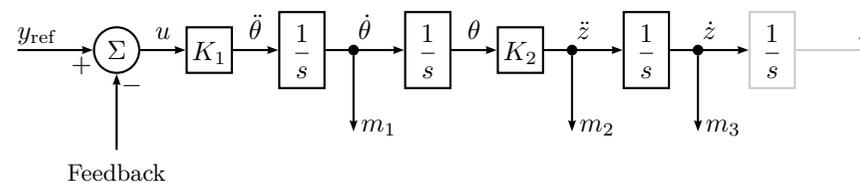


Figure 9.2c

b) In order to make the control duty of the astronaut easier we change the dynamics of the module by making internal feedback. The following signals are measurable:

$m_1$ , the attitude angular velocity measured using rate gyro.

$m_2$ , the acceleration in  $z$ -direction measured using accelerometers positioned on gyro-stabilized platforms.

$m_3$ , the velocity in  $z$ -direction measured using doppler-radar.

Calculate a state-feedback using these signals such that the closed loop system obtains its poles in  $s = -\frac{1}{2}$  and the control signal of the astronaut becomes the reference signal of the velocity in  $z$ -direction.

c) Suppose we by safety reasons are interested in the possibility of controlling the module even if the sensors measuring  $m_1$  and  $m_2$  are not working. Design a controller that can handle this and has approximately the same behavior as in a).

9.3 A DC motor with an external load,  $T$ , is described by

$$\begin{aligned}\omega &= \dot{\theta} \\ \dot{\omega} &= -\frac{1}{\tau}\omega + c_1 u + c_2 T\end{aligned}$$

where  $\theta$  is the angle,  $\omega$  the angular velocity,  $u$  the control signal,  $T$  the torque of the load, and  $c_1$ ,  $c_2$ , and  $\tau$  are constants.

a) Introduce a controller

$$u = l_0 \theta_{\text{ref}} - l_1 \theta - l_2 \omega$$

such that the poles of the closed loop system becomes  $\frac{1}{\tau}(-1 \pm i)$  and  $\theta = \theta_{\text{ref}}$  in steady-state if  $T = 0$  and  $\theta_{\text{ref}}$  is constant.

b) Introduce a modified controller

$$u = l_0 \theta_{\text{ref}} - l_1 \theta - l_2 \omega + u'$$

such that  $\theta = \theta_{\text{ref}}$  in steady-state even for constant non-zero  $T$  and constant  $\theta_{\text{ref}}$ .

9.4 A system can be described in state space form as

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \\ y &= (1 \quad -1) x\end{aligned}$$

We want to place the poles in  $\{-2, -3\}$ . Suggest an observer, and use a linear state feedback controller. Which are the poles of the closed loop system?

9.5 Is it possible to design an observer with poles in  $\{-5, -6, -7, -8\}$  for the system below? Motivate your answer.

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 10 \\ -3 \\ 2 \end{pmatrix} u \\ y &= (1 \quad 0 \quad 0 \quad 0) x\end{aligned}$$

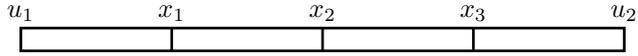


Figure 9.6a

9.6 We want to control the temperature in a long copper rod by heating or cooling its endpoints. Principally, this problem is described by a partial differential equation. To simplify the problem we assume that the temperature profile in the rod can be approximated by the temperatures  $x_1$ ,  $x_2$ , and  $x_3$  at three points. The temperatures in the end points are the inputs,  $u_1$  and  $u_2$ . All temperatures are relative to the temperature of the surroundings.

We get the following ordinary differential equations:

$$\begin{aligned}\dot{x}_1 &= \alpha(u_1 - x_1) + \alpha(x_2 - x_1) \\ \dot{x}_2 &= \alpha(x_1 - x_2) + \alpha(x_3 - x_2) \\ \dot{x}_3 &= \alpha(x_2 - x_3) + \alpha(u_2 - x_3)\end{aligned}$$

where  $\alpha$  is a constant that depends on the coefficient of thermal conductivity and the specific heat of the rod. For simplicity, let  $\alpha = 1$ . Consider the problem of controlling the temperature in  $x_1$ ,  $x_2$ , and  $x_3$  with  $u_1$  only, assuming  $u_2 = 0$ .

- Assume that we want to have an arbitrary temperature profile, that is, arbitrary values of  $x_1$ ,  $x_2$ , and  $x_3$ . Is this possible? Why/why not?
- Assume that all the temperatures  $x_1$ ,  $x_2$  and  $x_3$  are measurable. Find a state feedback that brings any initial state to zero as  $e^{-3t}$ .
- Assume that only one of the temperatures  $x_1$ ,  $x_2$ , or  $x_3$  is measurable, and that we still want a controller which damps a disturbance as  $e^{-3t}$  by using an observer. The sensor can be placed so that any of the three values  $x_1$ ,  $x_2$ , or  $x_3$  is measured. Which choices of measure point make it possible to control the system as desired? Give a motivation. Choose one of the points making the wanted design possible and design a controller, that is, an observer and a state feedback, giving the desired error damping.

9.7 Consider the lunar excursion module in Problem 9.2. Suppose that there are no rate-gyro measurements available but that the sensors measuring  $m_2$  and  $m_3$  are still working.

Show how  $m_1$  can be reconstructed from  $u$  and measurements of  $m_2$  such that the reconstruction error decreases arbitrarily fast without differentiation of any of the measured signals.

Propose a filter and describe the resulting controller when the feedback consists of both measured and reconstructed states as in Problem 9.2.

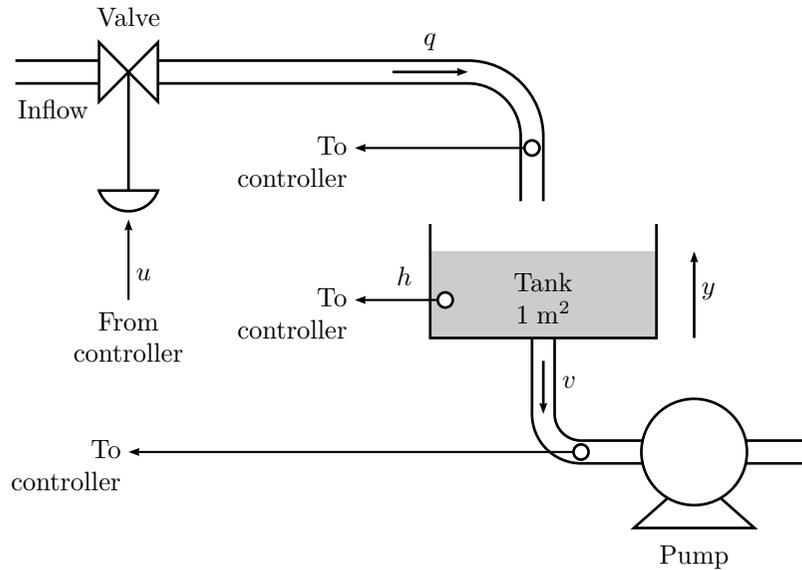


Figure 9.8a

9.8 In Figure 9.8a a level control system for a tank is shown. The objective is to keep the level at a desired value. Let  $u$ ,  $h$ ,  $q$ , and  $v$  denote small variations around the desired working point. The inflow,  $q$ , to the tank is determined by the valve,  $u$ , calculated by the controller. The outflow  $v$  is determined by the pump and deviations from zero is considered as process noise. The valve has some dynamics, which is modeled with the transfer function

$$Q(s) = \frac{k_1}{1 + Ts} U(s)$$

where  $k_1 = 1$  and  $T = 0.5$ . The level is given by

$$A\dot{h} = q - v$$

where the tank cross-section area is  $A = 1\text{m}^2$ .

- Let  $q$  and  $h$  be state variables and give a corresponding state space model of the process. Compute a state feedback  $u = -l_1q - l_2h + r$ , such that the closed loop system poles both are at  $-2$ .
- How large is the steady-state level error for a constant disturbance  $v = 0.1$  if  $r = 0$ ?

- Consider the closed loop system in a) and compute a feedforward control law from  $v$  to  $r$  such that the influence from  $v$  is completely eliminated. Exclude all terms in the control law in which  $v$  is differentiated to make it implementable. How does this modified feedforward control law work? Steady-state level error?
- Suppose that  $k_1$  differs slightly from 1, but that the same control law as in c) is used (the control law derived under the assumption  $k_1 = 1$ ). What happens with the steady-state level error?
- Propose a modified control law such that the stationary level error is zero for constant disturbances regardless of small deviations from the nominal value of  $k_1$ .

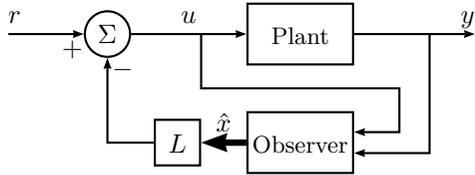


Figure 9.9a

9.9 We want to control the system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

with a state feedback. (This can be interpreted as a moving vehicle in one dimension, where  $x_1$  is the position,  $x_2$  is the velocity, and the acceleration is the control signal.) We introduce the control law

$$u(t) = -L\hat{x}(t) + r(t)$$

where  $\hat{x}$  is constructed by an observer

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

We choose the vectors  $K$  and  $L$  as  $L = \begin{pmatrix} 1 & 2 \end{pmatrix}$  and  $K^T = \begin{pmatrix} 4 & 4 \end{pmatrix}$ . These choices put the eigenvalues of  $A - BL$  in  $-1$  and the eigenvalues of  $A - KC$  in  $-2$ . A block diagram of the closed loop system is shown in Figure 9.9a. Due to a time delay, the real input is given by the equation

$$u(t) = -L\hat{x}(t - T) + r$$

What is the largest possible time delay  $T$  without the closed loop system getting unstable?

9.10 One wants to construct an observer for the system

$$\dot{x}(t) = \begin{pmatrix} -1 & 1 \\ a & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 2 & 1 \end{pmatrix} x(t)$$

- Suppose  $a = 1$ . Construct an observer with the poles in  $\{-5, -10\}$ . For which values of  $a$  is this possible?
- Suppose that the measured signal  $y$  is given by

$$y(t) = \begin{pmatrix} 2 & 1 \end{pmatrix} x(t) + v(t)$$

Here  $v(t)$  is the measurement noise. Compute the transfer function from  $v$  to  $\tilde{x}_1(t)$ , that is, the first element in the state vector for the observer error  $\tilde{x}(t) = x(t) - \hat{x}(t)$ .

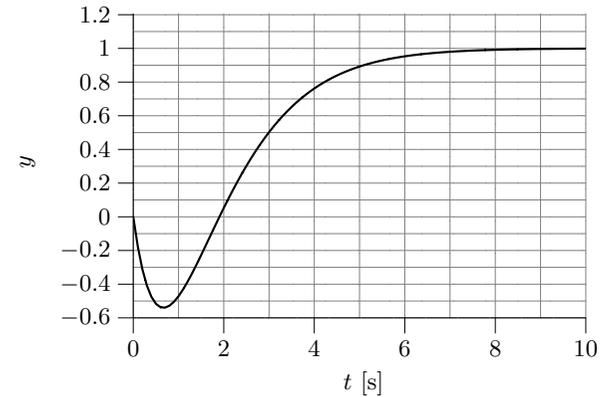


Figure 9.11a

9.11 Consider the system

$$G(s) = \frac{1 - s/\alpha}{(1 + s/\beta)^2}, \quad \alpha > 0, \beta > 0$$

Systems of this kind, that is, with a zero in the RHP have the property that the step response goes in the “wrong direction” initially, see Figure 9.11a.

- Show that the derivative of the step response at  $t = 0$ , that is,  $\dot{y}(0)$ , decreases as the zero of the system approaches the origin.
- Is it possible to use state feedback to eliminate the problem that the step response goes in the wrong direction initially? Justify your answer.

9.12 Theoretically one can place the poles of a controllable system arbitrarily. Which practical difficulties limit the performance that one can actually achieve?

9.13 Consider the system

$$\dot{x} = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} -1 \\ 2 \end{pmatrix} u$$

$$y = (1 \quad 1) x$$

- Determine  $L$  of a state feedback  $u = r - Lx$ , that places the poles at  $-2 \pm i$ .
- The state feedback of a) is used. It is observed that the output  $y(t) = 0$  for all  $t$  is obtained for a reference signal of the form  $r(t) = e^{\alpha t}$ . For what value(s) of  $\alpha$  does this occur?

9.14 Consider the model of a DC-motor

$$Y(s) = G(s)U(s)$$

where

$$G(s) = \frac{1}{s(s+1)}$$

- Generate a state space representation using MATLAB. Which physical signals are represented by the states?
- Suppose that the system is going to be controlled using state feedback

$$u(t) = -Lx(t) + l_0 r(t)$$

Compute the gain vector  $L$  and simulate the closed loop system for the following two choices of closed loop poles:

- ◊ Real poles at  $\{-2.2, -2.1\}$
- ◊ Poles at  $-1 \pm i$

Also compute  $l_0$  such that the closed loop system gets static gain one. In particular look at the properties of the step response and the magnitude of the control signal in the two cases. Which pole locations give the best trade off between response speed and control signal magnitude?

- Now let  $L$  be computed using linear quadratic optimization (LQ) in order to minimize the cost function

$$\int_0^{\infty} x(t)^T Q x(t) + u(t)^2 dt$$

for the three choices of weight matrices given below. Compute the closed loop poles and the step responses of the closed loop system for the three cases. Describe how the properties of the step responses in the different cases.

$$(i) \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(ii) \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix}$$

$$(iii) \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}$$

- Start from case (ii) and increase the weight on the control signal gradually until the cost function becomes

$$\int_0^{\infty} x(t)^T Q x(t) + 10 u(t)^2 dt$$

Compare the result with the result obtained for case (i).

- Start from case (i) and introduce a weight on the velocity  $\dot{y}(t)$ . Increase the weight gradually and study how the poles and the step response of the closed loop system change.

9.15 The ingestion and metabolism of a drug in a human body can be described by the following equations:

$$\begin{aligned}\frac{dq(t)}{dt} &= -k_1q(t) + u(t) \\ \frac{dm(t)}{dt} &= k_1q(t) - k_2m(t)\end{aligned}$$

where the input signal  $u(t)$  is the ingestion rate of the drug, the output  $y(t)$  is the mass  $m(t)$  of the drug in the blood, and  $q(t)$  is the mass of the drug in the gastrointestinal tract. The constants  $k_1$  and  $k_2$  are metabolism rates, satisfying  $k_1 > k_2 > 0$ .  $k_2$  characterizes the excretory process of the individual. In this example,  $k_1 = 0.05$  and  $k_2 = 0.02$ .

- Is the system controllable?
- Design a state feedback that places the closed loop poles in  $-0.1$ .

$q(t)$  (the mass of the drug in the gastrointestinal tract) cannot be measured, so to be able to use the state feedback in b) we need an observer.

- How should the poles of the observer be selected?
- Design an observer with poles in  $-0.2$ .

9.16 A system is described by the state space equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t - \tau) \\ y(t) &= \kappa Cx(t) + e(t)\end{aligned}\tag{9.1}$$

where it is known that

$$\begin{aligned}\tau &< 0.3 \\ 0.9 &\leq \kappa \leq 1.1\end{aligned}$$

It is also known that  $e$  is a sinusoidal disturbance with angular frequency 10 rad/s. The control design is based on the simplified model

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{9.2}$$

The specifications for the control system are given by:

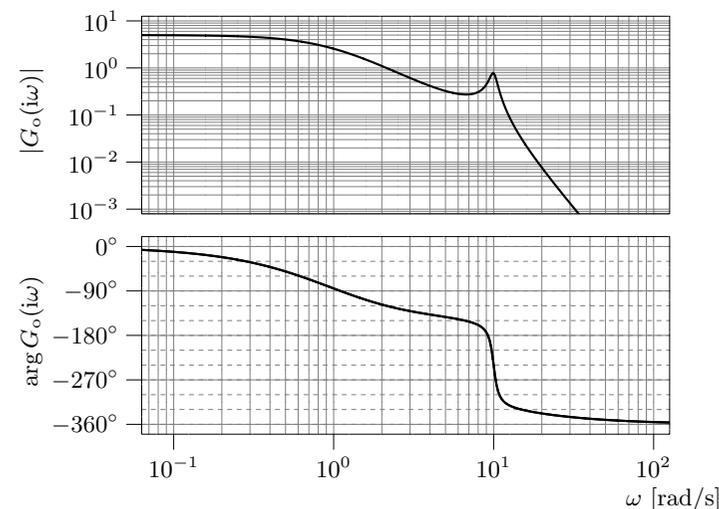


Figure 9.16a. Bode plot of the loop gain.

- The bandwidth of the closed loop system must fulfill  $\omega_B \geq 5$  rad/s.
- The closed loop system shall be stable despite the disturbance and the uncertainties in the parameters  $\tau$  and  $\kappa$ .
- The static gain of the closed loop system shall be 1 despite the disturbance and the uncertainties in the parameters  $\tau$  and  $\kappa$ .
- The closed loop system shall handle the measurement disturbance sufficiently well.

The regulator design is carried out using state space methods. The poles of the closed loop system, that is, eigenvalues of  $A - BL$ , are placed in  $\{-4, -2 \pm 2i\}$  and the poles of the observer, that is, the eigenvalues of  $A - KC$ , are placed in  $\{-17, -0.2 \pm 10i\}$ .

The figures below show different aspects of the control system. It is important to note that the diagrams are obtained using the model (9.2) and the designed regulator. Determine if the requirements 1–4 are fulfilled when controlling the system given by (9.1).

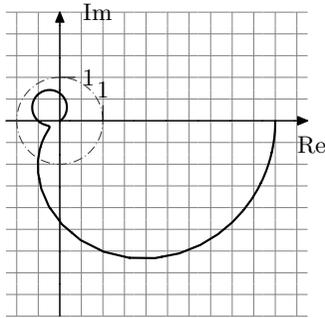


Figure 9.16b. Nyquist curve of the loop gain.

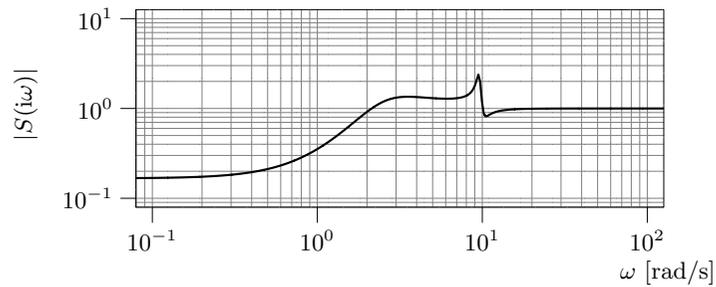


Figure 9.16c. Gain curve of the sensitivity function.

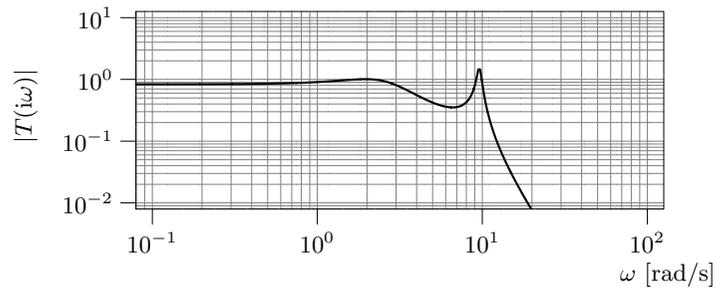


Figure 9.16d. Gain curve of the complementary sensitivity function.

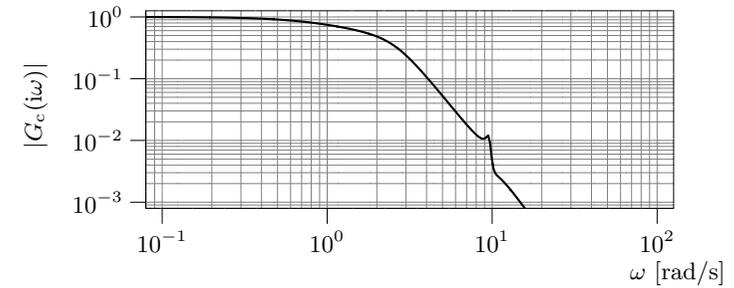


Figure 9.16e. Gain curve of the transfer function of the closed loop system.

9.17 In purification processes sometimes bacteria are used to consume the unwanted substance (possibly converting it to something more useful). Let  $\xi$  denote the amount of bacteria,  $\eta$  the amount of substance to be removed, and  $q$  the input flow (that contains the substance to be removed). The system is then described by a set of nonlinear differential equations

$$\begin{aligned}\dot{\xi} &= f_1(\xi, \eta, q) \\ \dot{\eta} &= f_2(\xi, \eta, q)\end{aligned}$$

When considering small deviations from an equilibrium the equations can be approximated by a linear system. In this example, the numeric values of the linearization are given by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u$$

where  $x_1$ ,  $x_2$ , and  $u$  denote the deviations from the equilibrium values of  $\xi$ ,  $\eta$ , and  $q$  respectively.

- Assume that both  $x_1$  and  $x_2$  can be measured. Determine a state feedback placing the closed loop poles in  $\{-2, -4\}$ .
- Assume that only  $x_2$  is measured. Is it possible to calculate the amount of bacteria  $x_1$  from this measurement if  $u$  is known? If the answer is yes: Why is it important that  $u$  is known?
- Assume that the value of  $u$  is unknown, but let it be known that it is constant. Is it then possible to calculate the amount of bacteria,  $x_1$  from a measurement of  $x_2$ ? In case it is, show some way of doing the computation.

9.18 A certain species of bacteria grows by consuming glucose, whose inflow is controlled. The following model is used

$$\dot{m} = (f - 1)m \quad \dot{f} = -m + q$$

where  $m$  is the amount of bacteria,  $f$  the amount of glucose and  $q$  the inflow of glucose. One wants the system to operate in the neighborhood of the operating point  $m = 1, f = 1, q = 1$ . Using the notation  $x_1 = m - 1, x_2 = f - 1, u = q - 1$ , an approximate model ( $x_1 x_2$  neglected) is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u \end{aligned}$$

Sometimes there are disturbances that are modeled as a constant, unknown external signal  $w$ :

$$\begin{aligned} \dot{x}_1 &= x_2 + w \\ \dot{x}_2 &= -x_1 + u \end{aligned}$$

One wants to drive  $x_1$  to a reference value  $r$  using  $u$  as control variable.

- $x_1$  is measured. What performance (stationary error, speed of response) can be obtained using a P controller?
- $x_2$  is measured. What performance (stationary error, speed of response) can be obtained by using a control law where  $u$  depends linearly on  $r$  and  $x_2$  if  $w = 0$ ? What happens when  $w \neq 0$ ?
- $x_2$  is measured. Determine a controller that does not differentiate  $x_2$ , gives an asymptotically stable closed loop system and makes  $x_1$  converge to  $r$  asymptotically for an arbitrary constant but unknown  $w$ .

### 9.19 2010-12-14 Uppg. 2

Ett system har modellerats genom att dela upp det i flera delsystem enligt Figur 9.19a.

- Finn överföringsfunktionerna:
  - $G_X(s)$ , så att  $X(s) = G_X(s)U(s)$ .
  - $G(s)$ , så att  $Y(s) = G(s)U(s)$ .

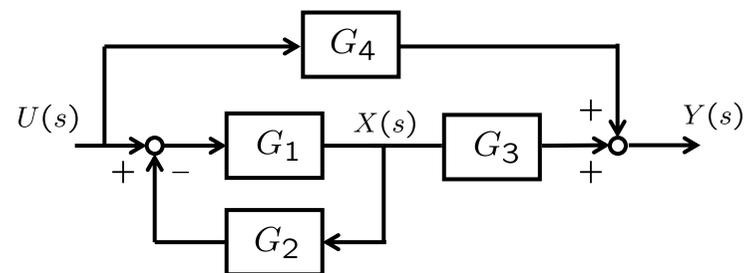


Figure 9.19a

- Genom att sätta samman överföringsfunktionerna i Figur 9.19a har man funnit att

$$Y(s) = G(s)U(s) \quad \text{där} \quad G(s) = \frac{s+2}{s^2}.$$

Skriv systemet  $G(s)$  på en valfri tillståndsform

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned}$$

Det vill säga, bestäm matriserna  $A, B$ , och  $C$ . Är din valda tillståndsform en minimal realisation? (Motivera ditt svar.)

- Bestäm en tillståndsåterkoppling för systemet ovan,

$$u(t) = -Lx(t) + l_0 r(t),$$

så att slutna systemets poler hamnar i  $\{-1, -1\}$  och så att det slutna systemets statiska förstärkning från  $r(t)$  till  $y(t)$  blir 1.

(Om du inte kunde svara på uppgift b), gör själv ett lämpligt val av  $A$  och  $B$ .)

- Designa en observerare för systemet ovan så att observerarens egenvärden hamnar i  $\{-10, -10\}$ .

(Om du inte kunde svara på uppgift b), gör själv ett lämpligt val av  $A$  och  $C$ .)

# 11 Implementation

11.1 If you “translate” the compensator

$$U(s) = KN\left(\frac{s+b}{s+bN}\right)E(s)$$

with Tustin’s formula you get a controller of the form

$$u(t) = \beta_1 u(t-T) + \alpha_1 e(t) + \alpha_2 e(t-T)$$

What are the values of  $\alpha_1$ ,  $\alpha_2$ , and  $\beta_2$ , if  $T = 0.1$ ,  $N = 10$ ,  $b = 0.1$ , and  $K = 2$ ?

11.2 Consider the system

$$\dot{y}(t) = u(t)$$

Suppose it is controlled with a computer, so that the control signal is constant over the sampling interval, that is,

$$u(t) = u_k, \quad kT \leq t < (k+1)T$$

- Introduce the notation  $y_k = y(kT)$  and derive a relation between  $y_{k+1}$ ,  $y_k$ , and  $u_k$ .
- Suppose we use the proportional feedback

$$u_k = -Ky_k$$

and that  $y(0) = y_0$ . What are the values of  $K$ , for which the closed loop system is stable?

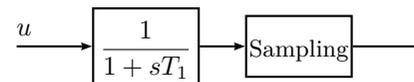


Figure 11.3a

11.3 Consider the system in Figure 11.3a, which illustrates sampling with prefiltering. Suppose we are sampling with the sampling period  $T$  and that  $u = u_0 + u_1$ , where  $u_0$  is an “interesting” low frequency signal in the frequency interval  $0 < \omega < \pi/T$  and that  $u_1$  is a sinusoidal control signal

$$u_1(t) = \sin \omega_2 t, \quad \frac{\pi}{T} < \omega_2 < \frac{2\pi}{T}$$

Since the sampling causes aliasing, the output will be

$$y(t) = y_0 + y_1$$

where  $y_0$  is interesting and  $y_1$  is a disturbance signal

$$y_1(kT) = A \sin(\omega_1 kT + \varphi), \quad \omega_1 < \pi/T$$

- What are  $A$ ,  $\omega_1$  and  $\varphi$ ?
- It is clear from a) that the choice of  $T$  affects the amplitude of the disturbance signal  $y_1$ . What is the smallest amplitude you can get if you do not want to damp any frequencies in  $u_0$  more than  $\sqrt{2}$  times?

11.4 **2011-01-13 Upg. 1d**

Genom att approximera en PI-regulator med Euler bakåt med samplingsintervall  $T = 1$  fås den tidsdiskreta regulatorn

$$u(t) = u(t-1) + 2e(t) - e(t-1)$$

Ange parametrarna  $K$  och  $T_I$  för motsvarande tidskontinuerliga PI-regulator

$$u(t) = K \left[ e(t) + \frac{1}{T_I} \int_0^t e(\tau) d\tau \right]$$

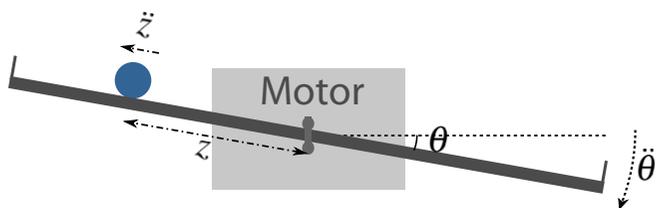


Figure 11.5a

### 11.5 2011-06-10 Uppg. 5

En kula är placerad på en bom enligt Figur 11.5a. En elektrisk motor kan rotera bommen och därmed få kulan i rullning. Kulans rörelse modelleras av differentialekvationen

$$\left(m + \frac{J}{r^2}\right) \ddot{z} = -mg \sin \theta + mz\dot{\theta}^2,$$

där  $m$  är kulans massa,  $J$  dess tröghetsmoment och  $r$  dess radie. Kulans position på bommen ges av  $z$  och bommens vinkel av  $\theta$ , enligt Figur 11.5a.

Bommens vinkelhastighet är proportionell mot den elektriska spänningen  $u$  över motorn, d.v.s.  $\dot{\theta} = Ku$ , där  $K$  är en konstant.

- a) i) Skriv ovanstående modell på tillståndsform med tillstånden  $\theta, z$  och  $\dot{z}$ , och
- ii) linjärisera sedan modellen kring en jämviktspunkt där  $u = \theta = \dot{z} = 0$ , och skriv ekvationerna på formen  $\dot{x} = Ax + Bu$ . Du kan anta att  $\frac{m}{m + J/r^2} = \frac{5}{7}$ .
- b) Vi fokuserar nu på reglering av enbart bommens vinkel  $\theta$ . Låt därför  $y = \theta$  vara utsignal. Om spänningen över motorn kommer från en D/A-omvandlare så är den konstant över varje samplingsintervall och ges av  $u(t) = u_k$ , för  $kT \leq t < (k+1)T$  där heltalet  $k$  anger sampelnummer och  $T$  samplingsintervallet.

Låt  $y_k = y(kT)$  vara den samplade utsignalen, och uttryck  $y_{k+1}$  i variablerna  $y_k$  and  $u_k$ .

- c) Antag att spänningen i Uppgift b) ges av  $u_k = -K_p y_k$ . För vilka  $K_p$  är det slutna systemet asymptotiskt stabilt? (Om du inte kunde lösa Uppgift b) kan du anta  $y_{k+1} = \frac{1}{2}y_k + \frac{1}{3}u_k$ .)



# Hints

This version: August 16, 2016



## 2 Dynamic Systems

2.1 Start with  $J\ddot{\theta} = -f\dot{\theta} + M$  and try to write  $M$  as a function of  $\theta$  and  $u$  using Kirchhoff's voltage law.

2.2 What is the relationship between the response of the system and the pole locations?

2.3 Separate the pure delay and the dynamic response. Use the final value theorem to find the steady state gain and calculate the time constant by estimating the time to reach 63% of the final value (neglecting the time delay).

2.4 Identify the coefficients  $\omega_0$  and  $\zeta$  in the system description

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0s + \omega_0^2}$$

2.9 See Glad&Ljung.

2.11 a) Consider what you can control, what is uncontrollable and what is desired.

b) Consider the relationship between the signals.

2.12 a) Use mass balance and assume that the densities are equal.

b) Consider the change in mass and change in component A.

c) Assume that all the other independent variables ( $q_1$ ,  $q_2$ ,  $c_{A,2}$ ) are constant.

2.13 a) Use mass and component balance.

## 3 Feedback Systems

- 3.1 a) Consider the three blocks; tank, valve, and PID. What is the input and the output from each block? Connect the blocks and consider  $v$  as a disturbance.

For the tank model use the fact that the flow into the tank is  $x - v$  and the amount of liquid changes as  $\dot{h} \cdot A$ .

- b) Consider the final value and the time constant.  
 d) Put  $F(s) = K$  and express the closed loop poles as a function of  $K$ .  
 e) Use the final value theorem.  
 f) Put  $F(s) = \frac{K_P s + K_I}{s}$  in the expression for the error, and use the final value theorem.

- 3.2 a) Use the expression for the poles from Problem 3.1.  
 b) Put  $F(s) = K_P + K_D s$  in the expression for the closed loop system from Problem 3.1. The relative damping is defined in Glad&Ljung.

- 3.3 Start with deriving an expression for the transfer function from the disturbance  $f_c$  to the error  $e$ .

- a) Use  $F(s) = K$  and the final value theorem.  
 b) Use  $F(s) = K_1 + K_2/s$  and the final value theorem.

- 3.5 a) The characteristic equation is

$$s(s+1)(s+3) + K(s+2) = 0$$

which gives  $P(s) = s(s+1)(s+2)$  and  $Q(s) = s+2$ .

- b) Characteristic equation:

$$s(s^2 + 2s + 2) + K = 0$$

$$P(s) = s(s^2 + 2s + 2), Q(s) = 1.$$

- c) Characteristic equation:

$$s(s-1)(s+6) + K(s+1) = 0$$

$$P(s) = s(s-1)(s+6), Q(s) = s+1.$$

- 3.6 Derive the general closed loop transfer function by first deriving the transfer function for the inner loop.

- a) Let  $\alpha = 0$ . The characteristic equation is then

$$s(s+2) + 4K = 0$$

Compute the poles explicitly as a function of  $K$ .

- b) The characteristic equation is

$$s(s+2) + 4K(1+s) = 0$$

- c) Characteristic equation:

$$s(s+2) + 4K(1+s/3) = 0$$

- d) Characteristic equation:

$$s^2 + 2s + 4 + 4\alpha s = 0$$

- 3.7 a) Derive the transfer function from  $\omega_{\text{ref}}$  to  $\omega$ .

- b) The characteristic equation is

$$(s+10)(s+4)(s-3) + 10K(s+1) = 0$$

- 3.8 a) Derive the closed loop transfer function by first deriving the transfer function for the inner loop. The characteristic function is

$$s((s+1)(s+10) + K_1) + K_2 = 0$$

We get two principally different root loci when there are complex starting points, and when all starting points are equal. Treat the cases separately.

3.9 a) The characteristic equation is

$$(s + 1)(s - 1)(s + 5) + K = 0$$

b) Characteristic equation:

$$(s + 1)(s - 1)(s + 5) + K(1 + 0.5s) = 0$$

3.10 a) Characteristic equation:

$$s^3 + 2s^2 + a(s^2 + 2s + 6) = 0$$

b) First check for which  $a$  the system is stable and the steady state requirement is fulfilled. Then use that “sinusoid in” gives “sinusoid out” after transients.

3.11 Check the root locus to find which  $K$ -values gives a stable/unstable system, more/less oscillative system.

3.12 Investigate the starting points and end points of the root locus.

3.13 Find the open loop transfer function and use the Nyquist criterion.

3.14 Since  $G(s)$  has no poles in the RHP, the closed loop system is stable if the Nyquist path of  $KG_o$  does not encircle  $-1$ . Note that the  $K$  will only modify the distance to the origin, not the shape of the curve.

3.15 Study the amplitude and phase of  $G(i\omega)$ .

3.16 a) Draw the complete Nyquist path and use the Nyquist criterion. (Note that  $G_o(-i\omega)$  is the mirror image of  $G_o(i\omega)$ , mirrored in the real axis.)

b) Use the final value theorem, and that  $G_o(0)$  is known from the Nyquist path.

c) Apply the Nyquist criterion to  $\frac{K}{s}G(s)$

3.17 The system oscillates when the open-loop gain is equal to  $-1$  (check  $Ke^{-i\omega T}G(i\omega)$ ).

3.18 Try to find the  $\omega$  that gives  $\arg F(i\omega)G(i\omega) = -180^\circ$ .

3.19 The Nyquist curve for small  $\omega$  determines if the system may have an integrator or not. Also check if the system is unstable for some  $K$  (from the Nyquist diagram).

3.24 Check the steady state error, the relative damping, etc.

3.25 a) To compute the closed loop transfer function combine

$$\theta(s) = G(s)U(s)$$

and

$$U(s) = F(s)(\theta_{\text{ref}}(s) - \theta(s))$$

b) The control error can be computed using

$$E(s) = \frac{1}{1 + F(s)G(s)}\theta_{\text{ref}}(s)$$

To find the steady-state error, use the final value theorem.

c) See b).

3.30 a) Consider what you can control, what is uncontrollable and what is desired.

b) Use mass and energy balance for both the tank and the heating system.

3.31 a) Compute the poles of the system.

b) Note that the system has negative sign in the numerator.

3.32 Check the pole for the closed loop system.

3.33 Use Newton's force equation  $F = ma$  to derive the transfer function for the astronaut.

# 4 Frequency Description

4.1 Determine the angular frequency  $\omega$  of the signals using the figure. Use the relationship saying that when  $u(t) = A \sin \omega t$  the output becomes

$$y(t) = |G(i\omega)| A \sin(\omega t + \arg G(i\omega))$$

to determine  $|G(i\omega)|$  and  $\arg G(i\omega)$ .

4.2 a) For  $K = 0.5$  the open loop system is given by

$$G_o(s) = F(s)G_r(s)G_s(s) = \frac{0.05(1 + s/0.02)}{s(1 + s/0.01)(1 + s/0.05)(1 + s/0.1)}$$

Use the rules in Glad&Ljung to make the Bode plot.

b) What can be said about the phase and gain margin when the output of the closed loop system oscillates with constant amplitude?

c) When the reference signal is  $A \sin \alpha t$  the output signal becomes

$$y(t) = |G_c(i\omega)| A \sin(\alpha t + \arg G_c(i\omega))$$

The Bode plot of the open loop system can be used to compute  $G_c(i\omega)$ .

4.3 a) Check the behavior of  $G(i\omega)$  when  $\omega \rightarrow 0$  and  $\omega \rightarrow \infty$  respectively. See also if the absolute value and the argument decrease monotoneously or not.

b) Translate the behavior of the amplitude and phase curves to a pole-zero diagram.

4.4 Check the final values of  $y(t)$  against the static gain  $G(0)$ . Check also the overshoots of  $y(t)$  against the height of the resonance peaks in  $G(i\omega)$ . Check the frequency of the oscillation in  $y(t)$  against the resonance frequency in  $G(i\omega)$ .

4.5 Use MATLAB, in particular the command `bode`.

4.6 Recall that for stable, linear systems “a sinusoid in gives a sinusoid out” after initial transients.

4.7 Recall that for stable, linear systems “a sinusoid in gives a sinusoid out” after initial transients.

4.11 a) Use the rules in Glad&Ljung to make the Bode plot.

b) What can be said of the phase and gain margin when the output of the closed loop system oscillates with constant amplitude?

4.12 a) What is the stability criterion in the Bode plot?

b) What is the current phase margin? Is a lead really necessary?

# 5 Compensation

- 5.1 Try a lead-lag design.
- 5.2 a) Glad&Ljung gives a good description of how Bode plots can be drawn by hand.  
b) A proportional controller does not affect the phase curve.  
c) Try lead compensator.
- 5.3 a) Draw asymptotic Bode plot (see 5.1) by hand or use MATLAB.  
b) Start with calculating the controller and then use the final value theorem.  
c) Try a lag compensator.
- 5.4 Start with drawing a Bode plot for the open loop transfer function. The final value theorem is a good tool in this exercise.
- 5.5 Check for signs of dominating poles, pure integrations, resonance frequencies...
- 5.6 Draw asymptotic Bode plot using the guidelines in Glad&Ljung. See the discussion on lead-lag compensators in Glad&Ljung.
- 5.7 Use values of  $|G(i\omega)|$  and  $\arg G(i\omega)$  to plot the Nyquist curve  $G(s)$ .
- 5.8 The time delay alters the phase curve but not the amplitude curve. Use the Nyquist stability criterion.
- 5.9 Check steady state level and rise time. Modify Figure 5.9b using  $G_A(s)$  and adapt a lead-lag compensator. You can use two lead compensators to achieve a big phase advance.
- 5.10 Start by adjusting Figure 5.10b to obtain the Bode plot of  $G$ .
- 5.11 b) It is possible to derive limits on  $K$  using either the Bode plot or the Nyquist curve.
- c) Use the final value theorem.  
d) A time delay is described by the transfer function  $e^{-sT}$ .
- 5.12 Think of all possible phase curves, for example originating from time delays, and think about the corresponding Nyquist curves or Bode plots.
- 5.13 See “Introduktion till CSTB” and previous exercises in this section.
- 5.14 Try a lead-lag compensator.
- 5.17 Study  $G_O(i\omega)$  and  $G_C(i\omega)$  at low frequencies and around the cross-over frequency of  $G_O(i\omega)$ .
- 5.20 c) What *open-loop* property has to be doubled to make the *closed-loop* system twice as fast?  
d) Look at the phase margin.  
e) What *open-loop* property do we have to maintain constant to have the *closed-loop* system approximately with the same overshoot?  
h) Can it disturb our previously designed phase margin?

## 6 Sensitivity and Robustness

6.1 The sensitivity function is the transfer function from  $v$  to  $y$ .

6.2 Derive the relative model error

$$G_{\Delta}(s) = \frac{G^0(s) - G(s)}{G(s)}$$

Make a simple plot of  $G_c(i\omega)$  using the information in the problem formulation. Compare with the inverse of the relative model error.

6.3 Convert the condition that the amplitude of  $y$  is larger than the amplitude of  $v$  to the condition

$$|1 + G_o(i\omega)| < 1$$

What does this inequality say about the distance between the Nyquist curve and the origin?

6.4 Compute the transfer function of the closed loop system. Apply the robustness criterion using the given upper bound of the relative model error.

6.5 a) Derive the relative model error

$$G_{\Delta}(s) = \frac{G^0(s) - G(s)}{G(s)}$$

and plot  $1/|G_{\Delta}(i\omega)|$ .

b) Determine the level that  $|G_c(i\omega)|$  cannot exceed.

6.6 a) Use the robustness criterion and check the condition for  $\tilde{G}(i\omega)$  when  $\omega \rightarrow \infty$ .

b) The characteristic equation of the closed loop system is

$$s^2(2 + 25\alpha) + 5s(2 + 25\alpha) + 25 = 0$$

6.7 a) The characteristic equation becomes

$$s^2(s + 1) + \alpha(s^2 + s + 4) = 0$$

b) Derive the relative model error

$$G_{\Delta}(s) = \frac{G^0(s) - G(s)}{G(s)}$$

Check where the absolute value of the inverse of the relative model error intersects  $|G_c(i\omega)|$  given in the figure. It is sufficient to check the low frequency asymptote.

c) What can be said about the necessity and sufficiency of the stability conditions in a) and b)?

6.8 Check where the absolute value of the relative model error intersects  $|G_c(i\omega)|$  given in the figure.

6.9 Derive the closed loop equation relating  $y(t)$ ,  $r(t)$ ,  $v(t)$ , and  $n(t)$  using  $Y(s) = V(s) + G_o(s)(R(s) - N(s) - Y(s))$ . Then use the fact that the sensitivity function  $S(s)$  and the complementary sensitivity function  $T(s)$  are related as  $S(s) + T(s) = 1$ . (Here  $T(s)$  coincides with the closed loop system.)

6.10 a) The relative model error is given by

$$G_{\Delta}(s) = \frac{G^0(s) - G(s)}{G(s)}$$

b) Use MATLAB and results from previous exercises.

6.11 Recall that for stable, linear systems “a sinusoid in gives a sinusoid out” after initial transients.

6.12 Create the loop gain transfer function and use its Bode plot to check stability.

6.13 The sensitivity function is the transfer function from  $v$  to  $y$ .

## 7 Special Controller Structures

- 7.1 a) Derive the transfer function from  $w$  to  $\theta_m$  which then implies the open loop transfer function

$$\Theta(s) = \frac{0.9}{(1 + s/0.033)(1 + s/0.33)(1 + s)}W(s)$$

Draw the Bode plot using the rules from Glad&Ljung.

- b) Draw the Bode plot using the rules from Glad&Ljung.

- 7.2 a) Use the relationship

$$H(s) = \frac{1}{As} \left( \frac{1}{1 + s/2} U(s) - V(s) \right)$$

- b) Derive the transfer function from  $V$  to  $H$  when both feedforward and feedback are used.

- 7.3 a) Use  $Y(s) = (G_u(s)F_f(s) + G_v(s))V(s)$ .

- b) Recall that for stable, linear systems “a sinusoid in gives a sinusoid out” after initial transients.

## 8 State Space Description

8.1 Define  $x_1 = \theta$ ,  $x_2 = \dot{\theta}$ , and utilize the differential equation for the motor.

8.2 For the nonlinear equation  $\dot{x}_2 = f_2(x_1, x_2, u)$ , the linearized equation is given by

$$\begin{aligned} \dot{x}_2 &= f_2(x_{1,0}, x_{2,0}, u_0) \\ &+ \frac{\partial f_2}{\partial x_1}(x_{1,0}, x_{2,0}, u_0) \cdot (x_1 - x_{1,0}) \\ &+ \frac{\partial f_2}{\partial x_2}(x_{1,0}, x_{2,0}, u_0) \cdot (x_2 - x_{2,0}) \\ &+ \frac{\partial f_2}{\partial u}(x_{1,0}, x_{2,0}, u_0) \cdot (u - u_0) \end{aligned}$$

8.3 Define  $x_1 = y$ ,  $x_2 = \theta$ , and  $x_3 = z$ . Use block diagram algebra to find expressions for  $s \cdot X_i(s)$ , then use the inverse Laplace transform.

8.4 Use canonical forms.

8.5 Take the Laplace transform of  $g(t)$ .

8.6  $G(s) = C(sI - A)^{-1}B$ .

8.7  $x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau$

8.8 a) Insert the control signals and take Laplace transforms. Use the final value theorem.

b) Examine the difference  $u_1(t) - u_2(t)$  for arbitrarily small constant  $\epsilon$ .

8.9 Check controllability.

8.10 The controllable subspace is spanned by the linearly independent columns of  $\mathcal{S}$ . The unobservable subspace is spanned by the null space of  $\mathcal{O}$ .

8.11 b) Compare what happens to the states as  $t \rightarrow \infty$ , to the transfer function poles.

c) Check if  $\det \mathcal{S}$  and  $\det \mathcal{O}$  are nonzero.

8.12 The system is minimal, compute the transfer function.

8.13 a) For small deviations around 0,  $\sin(\phi) \approx \phi$ ,  $\cos(\phi) \approx 1$ . Take  $\ddot{z}$  as input.

b)  $\det \mathcal{S} = \frac{1}{\alpha}(1 - \frac{1}{\alpha})^2$

8.15 a) Combine mass balance with the given equation for reaction speed.

8.17 a) Use  $\dot{x} = 0$  and  $u \equiv 1$ .

b)  $G(s) = C(sI - A)^{-1}B$

8.18 a) A stationary point fulfills  $f(x_0, u_0) = 0$ ,  $y = h(x_0, u_0)$ .

b) Use

$$\begin{aligned} A &= f_x(x_0, u_0) & B &= f_u(x_0, u_0) \\ C &= h_x(x_0, u_0) & D &= h_u(x_0, u_0) \end{aligned}$$

## 9 State Feedback

- 9.1 a) The closed loop system  $\dot{x} = Ax + Bu$ ,  $y = Cx$ ,  $u = -Lx + y_{\text{ref}}$  has characteristic polynomial  $\det(sI - A + BL) = 0$ .
- b) The observer poles are given by  $\det(sI - A + KC) = 0$  and should be placed to the left of the closed loop poles.
- 9.2 a) Write the system in state space form by introducing three state variables corresponding to the outputs of the three left-most integrators in the figure ( $\dot{z} = \text{output}$ ). Design a state feedback controller  $u = -Lx + y_{\text{ref}}$  and place the poles in  $-0.5$ .
- c) Design an observer with poles to the left of the closed loop poles.
- 9.3 a) The constant  $l_0$  can be found by using that  $\dot{\theta} = \dot{\omega} = 0$  at steady state.
- b) Introduce the integrated control error as an auxillary state.
- 9.4 Decompose the system into two subsystems, one controlled by  $u_1$  and one by  $u_2$ , and check the controllability.
- 9.5 Is the system observable?
- 9.6 Is the system observable?
- 9.7
- Is the system controllable?
  - $\tilde{x}(t)$  converges to zero as  $p(t)e^{-\alpha t}$  if  $(A - KC)$  has a double eigenvalue in  $-\alpha$ .
  - Check the observability of the system.
- 9.9 Study the phase margin of the open loop system.
- 9.11 Use the initial value theorem.
- 9.12 Compute the transfer function from  $u(t)$  to  $z(t) = L\hat{x}(t)$ , that is, the loop gain, and check the stability margin given a certain time delay  $T$ .
- 9.13
- The closed loop poles are given by  $\det(sI - A + BL) = 0$
- 9.17 a) The closed loop system  $\dot{x} = Ax + Bu$ ,  $u = -Lx$  has the characteristic polynomial  $\det(sI - A + BL) = 0$ .
- b) Check observability.
- c) Introduce a new state.

# 11 Implementation

# Answers

This version: August 16, 2016



# 1 Mathematics

1.1 a) A step has Laplace transform  $\frac{A}{s}$ .

b) A ramp has Laplace transform  $\frac{A}{s^2}$ .

c)  $\frac{1}{s+2}$

d)  $\frac{s}{s^2+25}$

e)  $sU(s) - u(0)$

f)  $sU(s)$ . ( $u(0) = 0$  is a common assumption in the course.)

g)  $s^2U(s) - su(0) - \dot{u}(0)$

h)  $s^2U(s)$ . ( $u(0) = \dot{u}(0) = 0$  is a common assumption in the course.)

i) A time delayed signal has Laplace transform,  $e^{-sT}U(s)$ .

1.2 a)  $\lim_{t \rightarrow \infty} y(t) = 5/2$

b)  $Y(s) = \frac{1}{s+2}U(s)$

1.3 The general solution is given by

$$y(t) = C_1 e^{-2t} + (C_2 + C_3 t) e^{-t} - \frac{3}{100} (\cos(2t) + 7 \sin(2t))$$

1.4 a)  $y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$ ,  $t \geq 0$

b)  $y(t) = 1 - 0.5e^{-t} + 0.5 \sin t - 0.5 \cos t$

1.5 a)  $\sqrt{2}e^{i\frac{\pi}{4}}$

b)  $\frac{\sqrt{2}}{10} e^{-i\frac{105}{180}\pi}$

c)  $1 + \sqrt{3}i$

d)  $-5$

1.6

decibel (dB <sub>20</sub> )	Definition	Amplification $F$
20	$20 \log F = 20 \Rightarrow$	$F = 10^1 = 10$
-3	$20 \log F = -3 \Rightarrow$	$F = 10^{-3/20} \approx 0.708 \approx \frac{1}{\sqrt{2}}$
0	$20 \log F = 0 \Rightarrow$	$F = 10^0 = 1$
10	$20 \log F = 10 \Rightarrow$	$F = 10^{0.5} = \sqrt{10} \approx 3.16$
-10	$20 \log F = -10 \Rightarrow$	$F = 10^{-0.5} = \frac{1}{\sqrt{10}} \approx 0.316$

1.7 Multiplication of the two matrices gives the unit matrix.

1.8

$$\lambda_1 = 3 \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\lambda_2 = -1 \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\lambda_3 = 4 \quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

1.9

$$T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

1.10 A basis for the null space is for example

$$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

A basis for the range space is

$$\begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix}$$

The rank of the matrix is hence 3.

1.11 a)  $f(t) = 1 - e^{-t}; 1.$

b)  $f(t) = -0.5e^{-t} + 0.5e^t; \infty.$

c)  $f(t) = e^{-t} \cdot t; 0.$

1.12  $y^{(3)} + 2\ddot{y} + 2\dot{y} + y = u$

## 2 Dynamic Systems

2.1 a) Differential equation

$$\ddot{\theta} + \frac{1}{\tau} \cdot \dot{\theta} = k_0 \cdot u$$

where

$$\frac{1}{\tau} = \frac{R_a f + k_a k_v}{J R_a} \quad k_0 = \frac{k_a}{J R_a}$$

b) Transfer function

$$G(s) = \frac{\theta(s)}{U(s)} = \frac{k_0}{s(s + 1/\tau)}$$

c) Step response

$$\theta(t) = k_0 \tau t - k_0 \tau^2 (1 - e^{-t/\tau})$$

2.2 (1)  $K = 0.1$

(2)  $K = 2.5$

(3)  $K = 3$

(4)  $K = 0.5$

2.3  $G(s) = \frac{10e^{-(L/V)s}}{1+3s}$

2.4 a)  $a < 1$

b)  $b = 2$

2.5 A-B, B-F, C-A, D-C, E-E, F-D.

2.6

System	$T_r$	$T_s$	$M$	poles
$G_A$	3.3	4.7	0%	-1, -1
$G_B$	1.2	13.6	52%	$-0.2 \pm i0.98$
$G_C$	10.6	14.6	0%	-4.8, -0.2
$G_D$	1.7	5.4	16%	$-0.5 \pm i0.87$
$G_E$	0.8	2.6	16%	$-1 \pm i1.73$

2.7  $\alpha > 0$  gives overshoot,  $\alpha < 0$  gives undershoot.

2.8  $y(t) = \mathcal{L}^{-1}(G(s)\frac{1}{s})$

2.9 a) 1.5

b)  $M \approx 26\%$

c)  $T_r \approx 1.5$

d)  $T_s \approx 7.8$

2.10  $G_1$ -C,  $G_3$ -B,  $G_4$ -A,  $G_5$ -D.

2.11 a) The signals can be classified as

- ◇ Disturbances signal: Acid process flow (unknown pH and flow)
- ◇ Control signal: NaOH solution
- ◇ Measured and controlled signal: The pH of the outflow

b) A block diagram where the control strategy is based on feedback could look like Figure 2.11a

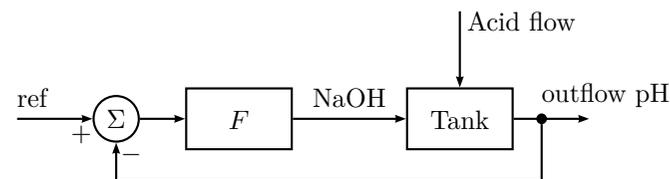


Figure 2.11a

2.12 a)  $q^* = 1.5 \text{ m}^3/\text{min}$  and  $c_A^* = 2.0 \text{ kmol/m}^3$ .

b) The model is nonlinear since the model is described by the following non-linear equations

$$\frac{d(\rho V)}{dt} = \rho(q_{in} - q_{out})$$

$$\frac{d(Vc_A)}{dt} = q_1 c_{A,1} + q_2 c_{A,2} - qc_A$$

c)  $k_0 = 2.0 \text{ kmol/m}^3$ ,  $k_1 = 0.13 \text{ kmol/m}^3$ , and  $\tau = \frac{1}{1.5} = 0.67 \text{ min}$ .

2.13 a) The model is given by

$$y_i = \frac{\alpha x_i}{1 + (\alpha - 1)x_i}$$

$$\frac{dM_i}{dt} = L_{i-1} + V_{i+1} - L_i - V_i$$

$$M_i \frac{dx_i}{dt} = L_{i-1}(x_{i-1} - x_i) + V_{i+1}(y_{i+1} - x_i) + V_i(x_i - y_i)$$

b) The linearized model is given by

$$M_i^* \frac{dx_{i\Delta}}{dt} = L_{i-1}^* x_{i-1,\Delta} + V_{i+1}^* y_{i+1,\Delta} - L_i^* x_{i\Delta} - V_i^* y_{i\Delta}$$

$$+ x_{i-1}^* L_{i-1,\Delta} + y_{i+1}^* V_{i+1,\Delta} - x_i^* L_{i\Delta} - y_i^* V_{i\Delta}$$

$$y_{i\Delta} = \frac{\alpha}{(1 + (\alpha - 1)x_i^*)^2} x_{i\Delta}$$

2.14 A -  $iv$ , B -  $i$ , C -  $ii$  and D -  $iii$ .

### 3 Feedback Systems

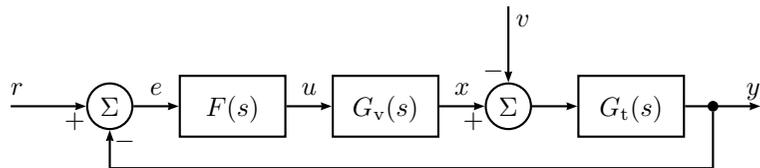


Figure 3.1a

3.1 a) Transfer function of the tank  $G_t(s) = \frac{1}{s}$ . Block diagram see Figure 3.1a.

b)  $k_v = 2, T = 5$

c)  $\frac{H(s)}{H_{ref}(s)} = \frac{G_t(s)G_v(s)F(s)}{1+G_t(s)G_v(s)F(s)}, \frac{H(s)}{V(s)} = -\frac{G_t(s)}{1+G_t(s)G_v(s)F(s)}$

d)  $K < 0.05$

e)  $\frac{1}{2K}$

f) 0

3.2 a)  $-0.1 \pm \sqrt{0.39}i$

b)  $K_D > 1.7$

3.3 a)  $-a/K$

b) 0

3.4 a) For small values of  $K_P$  the step response is slow, well damped and the steady state error is large. For increasing  $K_P$  the step response becomes faster but more oscillatory, while the error is reduced. For large  $K_P$  the amplitude of the oscillations increases, that is, the closed loop system becomes unstable.

b) The integrator in the regulator eliminates the steady state error. A too small value of  $K_I$  gives a large settling time while a too large value gives an oscillatory (finally unstable) closed loop system.

c) Using the (approximate) derivative of the error in the regulator increases the damping of the closed loop system. Increasing  $K_D$  too much, however, gives that an oscillation with higher frequency appears in the step response and finally (approximately when  $K_D > 65$ ) the closed loop system becomes unstable.

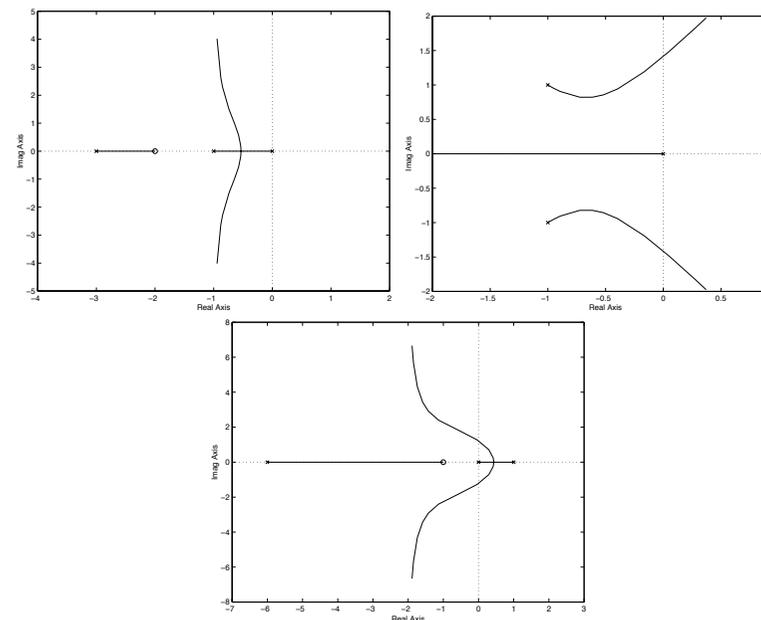


Figure 3.5a

3.5 Root loci are shown in Figure 3.5a.

b) Intersection with the imaginary axis for  $K = 4, \omega = \pm\sqrt{2}$ .

c) Intersection with the imaginary axis for  $K = 7.5, \omega = \pm\sqrt{1.5}$ .

Conclusions about the step response of the corresponding systems:

- a) Asymptotically stable all  $K > 0$ .  
 Small  $K$ : No oscillations, larger  $K$  gives faster system.  
 Larger  $K$ : Oscillations. Larger  $K$  gives more oscillations.
- b) Asymptotically stable for  $0 < K < 4$ . Oscillating all  $K > 0$ .  
 Small  $K$ : larger  $K$  gives faster system.  
 Larger  $K$ : larger  $K$  gives more oscillating system. Unstable for large  $K$  ( $> 4$ ).
- c) Asymptotically stable for  $K > 7.5$ . Unstable for  $K < 7.5$ . Stable and oscillating for  $K > 7.5$ . Larger  $K$  gives faster system, until the real pole becomes dominating, then larger  $K$  gives a slower system.

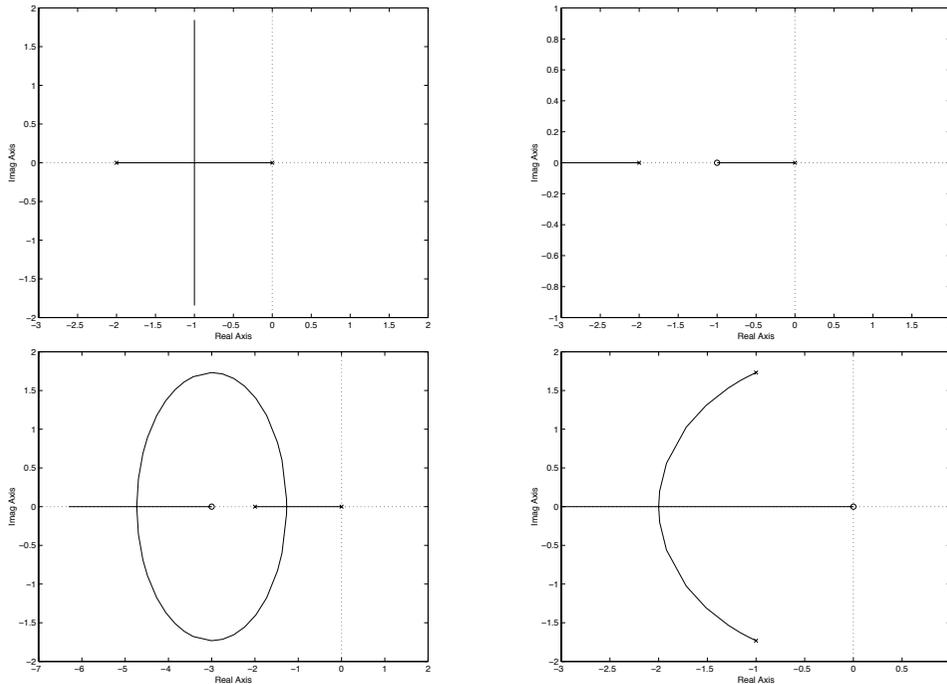


Figure 3.6a

### 3.6 General characteristic equation:

$$s(s + 2) + 4K(1 + \alpha s) = 0$$

The root loci are shown in Figure 3.6a.

- a) Asymptotically stable for all  $K > 0$ , oscillatory for large  $K$ .
- b) Asymptotically stable for all  $K > 0$ , not oscillatory for any  $K$ .
- c) Asymptotically stable for all  $K > 0$ , no oscillations for small and large  $K$ , faster for large  $K$ .
- d) Asymptotically stable for all  $\alpha > 0$ . Oscillatory for small  $\alpha$ . Larger  $\alpha$  gives more damped system.

With the tachometer feedback we can make the system both fast and well damped. The tachometer feedback is equivalent to the D-part in a PID controller.

- 3.7 a) The system is unstable so  $\omega$  will grow to infinity.
- b) The root locus is shown in Figure 3.7a. The system is asymptotically stable for  $K > 12$ .

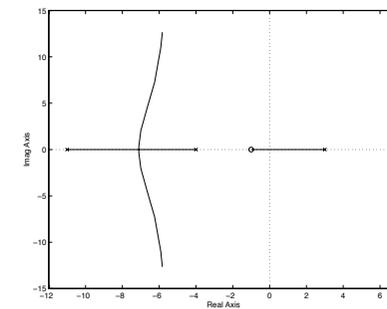


Figure 3.7a

- c) No. When  $K = 12$ ,  $s = 0$  is one pole but the other two are complex.

- 3.8 a) Starting points:  $s = -5.5 \pm \sqrt{5.5^2 - 10 - K_1}$ . The starting points are all real for  $K_1 \leq 20.25$ , while we have complex starting points for  $K_1 > 20.25$ . The two principal root loci are shown in Figure 3.8a. The system is asymptotically stable for  $0 < K_2 < 11K_1 + 110$ .

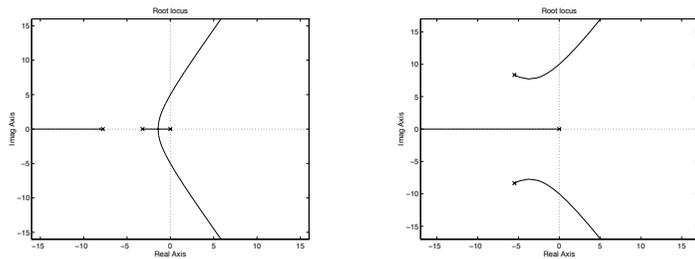


Figure 3.8a

- b) A larger  $K_1$  gives stability for larger  $K_2$ .

3.9 Root loci in Figure 3.9a.

- a) The system is unstable for all  $K$ .  
 b) Asymptotically stable for  $K > 5$ .

3.10 a) The root locus is shown in Figure 3.10a. The system is asymptotically stable for  $a > 1$ .

- b) The smallest amplitude is 0.1.

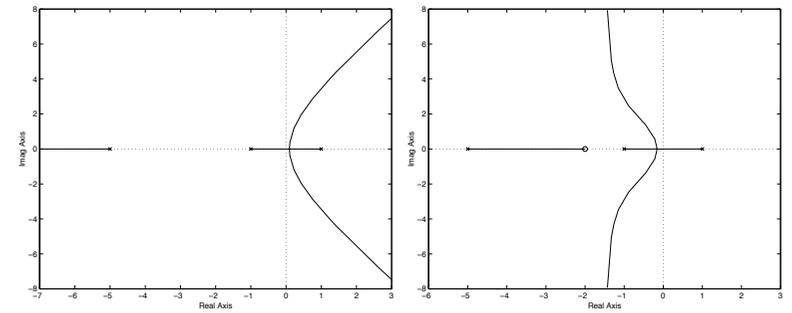


Figure 3.9a

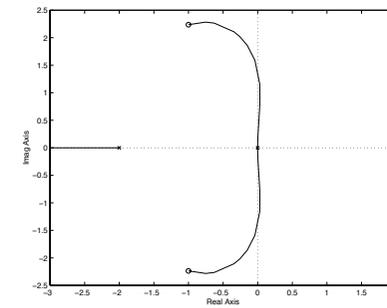


Figure 3.10a

3.11	$K$	Step
	4	C
	10	D
	18	B
	50	A

3.12 The poles of the system all tends to points in the LHP or to  $-\infty$  for large  $K$ .

3.13 The system is stable for  $K/A < \pi$ .

3.14 a) The closed loop system is stable in (i), (ii), and (iv).

- b) Stable when: (i)  $K < 2.5$ , (ii)  $K > 0$ , (iii)  $K < 1/2$ , and (iv)  $K < 1/4$  or  $K > 1/2$ .

3.15 The Nyquist curves are shown in Figure 3.15a.

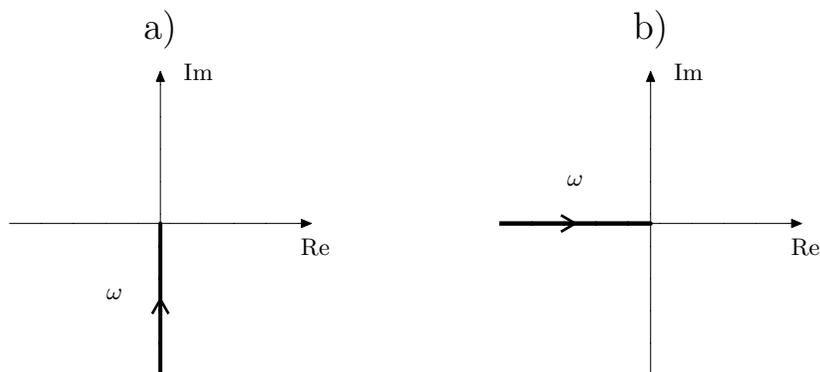


Figure 3.15a

3.16 a)  $K < 2/3$

b)  $\frac{1}{1+2K}$  when  $K < 2/3$

c)  $K < 2/3$

3.17  $\tau = 1.69$

$$T = \frac{\pi}{2} - \arctan \tau = 0.53$$

$$T_1 = \pi - 2 \arctan \frac{\tau}{2} = 1.74$$

3.18  $K < 2$

3.19 Root locus 1.

3.20 P  $\Rightarrow b_0 = b_2 = 0$   
 I  $\Rightarrow b_0 = b_1 = 0$   
 D  $\Rightarrow b_1 = b_2 = 0$

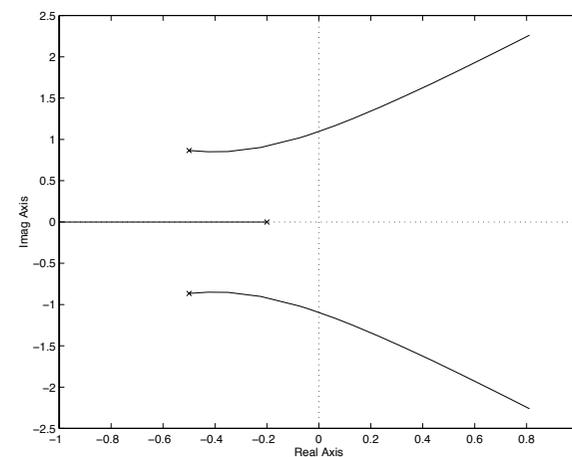


Figure 3.21a

3.21 a) The root locus with respect to  $K_P$  is shown in Figure 3.21a. When  $K_P$  increases the two complex poles move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for  $K_P \approx 6.2$ , the poles cross the imaginary axis and the closed loop system becomes unstable. This result is in accordance with Problem 3.4. For small values of  $K_P$  the properties of the step response are mainly determined by the real pole close to the origin. For larger values the complex poles start to dominate and when the complex poles cross the imaginary axis the amplitude of the oscillations in the step response increases and the system becomes unstable.

Note, however, that the root locus alone does not give sufficient information to tell how the steady state error changes with the parameter.

b) The root locus with respect to  $K_I$  is shown in Figure 3.21b. For small  $K_I$  the response of the closed loop system is dominated by the poles on the real axis close to the origin. When  $K_I$  increases the poles become complex and move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for  $K_I \approx 1.5$ , the poles cross the imaginary axis, that is, the closed loop system becomes unstable. As can be seen in Problem 3.4 a small value of  $K_I$ , that is, a pole close to the origin, gives a slow step response. When  $K_I$  increases the dominating

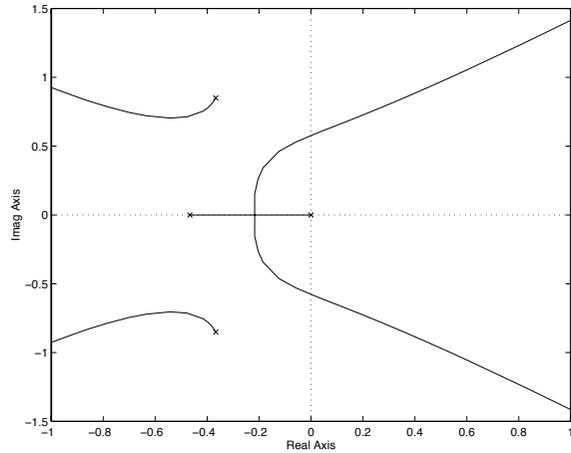


Figure 3.21b

poles become complex and the step response becomes oscillatory.

A large settling time will typically follow if the system is slow or has poor damping. Here, the large settling time for small  $K_I$  is due to the system being slow. That the steady state error is eliminated cannot easily be seen in the root locus.

c) The root locus with respect to  $K_D$  is shown in Figure 3.21c. When  $K_D$  increases the complex poles closest to the origin move towards the origin and at the same time the damping of the poles is increased. When  $K_D$  increases even more the second pair of complex poles moves towards the imaginary axis giving a high frequency oscillation which finally gives instability.

- 3.22 a) The Nyquist curve is “far away” from the point  $-1$  for all frequencies and the step response of the closed loop system is well damped. As  $K_P$  increases the Nyquist curve grows in size and for  $K_P = 6.2$  the Nyquist curve reaches  $-1$  and thus is the limit of stability.
- b) For low frequencies the Nyquist curve is now far away from the origin since the integrating part makes  $|G(i\omega)|$  large for low frequencies. The Nyquist curve now passes closer to  $-1$  which results in a more oscillatory closed loop system. The system becomes unstable around  $K_I = 1.44$ .

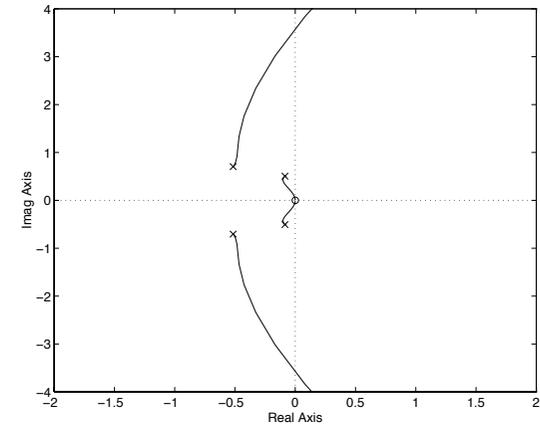


Figure 3.21c

c) The Nyquist curve is now further away from  $-1$  which corresponds to an improved damping of the closed loop system. The system becomes unstable around  $K_D = 66$ .

- 3.23 a)  $\omega_c = 0.38$ ,  $\omega_p = 1.1$ ,  $\varphi_m = 94^\circ$  and  $A_m = 3.1$ .
- b) The closed loop system is now much more oscillatory due to the reduced phase and gain margins.
- c)  $K_P = 3.1$ .

3.24 A-*iii*, B-*i*, C-*iv*, D-*ii*.

3.25 a)  $K_P$  small  $\Rightarrow$  Both poles on the real axis, but one pole very close to the origin  $\Rightarrow$  Slow but not oscillatory system.

$K_P = 1/(4\tau^2 k_0) \Rightarrow$  Both poles in  $-1/(2\tau)$ , that is, faster than in (1) but still no oscillations.

$K_P$  large  $\Rightarrow$  Complex poles with large imaginary part relative to the real part, that is, oscillatory system.

b) If the reference is a step,

$$\lim_{t \rightarrow \infty} e(t) = 0$$

If the reference is a ramp,

$$\lim_{t \rightarrow \infty} e(t) = \frac{A}{K_P k_0 \tau}$$

c)  $\lim_{t \rightarrow \infty} e(t) = 0$

3.26  $G_c = \frac{G_o}{1+G_o}$

3.27 a)  $G_o = FG$

b)  $G_c = \frac{FG}{1+FG}$

c)  $G_{ny} = -\frac{FG}{1+FG}$

d)  $G_{re} = \frac{1}{1+FG}$

3.28 a)  $\frac{3A}{3+K}$

b)  $F(s) = \frac{1}{s}$  (for example)

c) Poles in  $-2, -2$ . No zeros.

3.29 A-4, B-2, C-3, D-1.

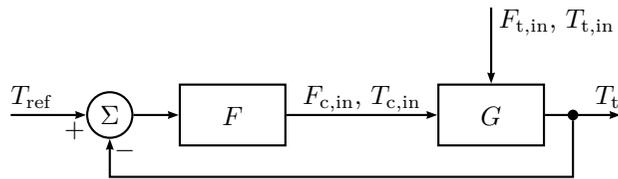


Figure 3.30a

3.30 a) See the block diagram in Figure 3.30a. There, the signals are classified as:

◇ Input  $F_{c,in}$  and  $T_{c,in}$

◇ Output  $T_t$

◇ Disturbance  $F_{t,in}$  and  $T_{t,in}$

b) The model is given by

$$V_t \frac{dT_t}{dt} = F_t(T_{t,in} - T_t) + \frac{U}{c_{t,in}^p \rho_t} (T_c - T_t)$$

$$V_c \frac{dT_c}{dt} = F_c(T_{c,in} - T_c) - \frac{U}{c_c^p \rho_c} (T_c - T_t)$$

c)  $T_{t\Delta}(s) = \frac{32}{(s+3.675)(s+0.185)} F_{c\Delta}(s)$

d,e) The root locus for a P controller is shown in Figure 3.30b.

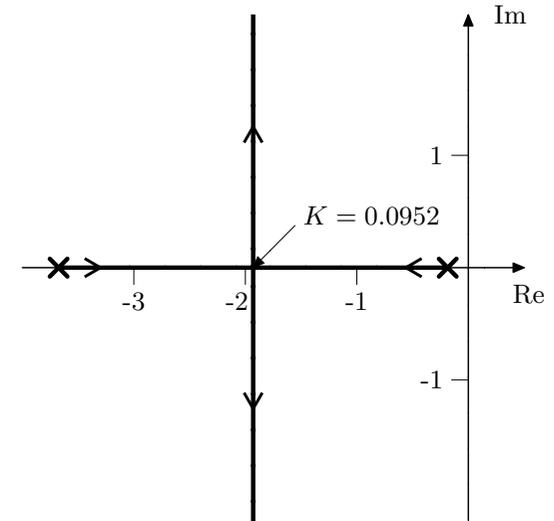


Figure 3.30b

3.31 a) The system has a pole in  $-3$ .

b) The system is stable for  $K \leq -3$ .

3.32  $K > \mu$

3.33  $K_2 < 1$  and  $K_1 = 200/K_2^2$ .

3.34

# 4 Frequency Description

4.1  $G(s) = \frac{0.16}{s+0.16}$

4.2 a) See figure in the solution.  $\omega_c = 0.025$ ,  $\varphi_m = 31^\circ$ ,  $A_m = 2.5$ .

b) The period time will be 108 seconds,  $K = 1.25$ .

c)  $B = 8^\circ$ ,  $\beta = 0.02$  rad/s and  $\varphi = -42^\circ$ .

4.3 a) Figure 4.3a in Solutions.

b) Figure 4.3b in Solutions.

4.4 A-B, B-C, C-D, D-A.

4.5 a)

System	$G(0)$	$\omega_B$	$\omega_r$	$M_p$
$G_A$	1	0.64		
$G_B$	1	1.5	1	2.5
$G_C$	1	0.21		
$G_D$	1	1.27	0.7	1.15
$G_E$	1	2.54	1.4	1.15

b) The bandwidth of a system is (approximately) inversely proportional to the rise time. The damping is inversely proportional to the height of the resonance peak. A large peak implies low damping and large overshoot.

4.6  $y(t) = \frac{1}{\sqrt{5}} \sin(2t - 1/2 - 4 - \frac{\pi}{2} - \arctan 2)$ .

4.7 a)  $0.45 \sin(2t - 1.1)$

b) Unstable system.

c)  $0.11 \sin(2t - 2.4)$

d)  $0.45 \sin(2t - 2.1)$

4.8 a,b)

$\omega$	$ G(i\omega) $	$\arg G(i\omega)$
1	1 = 0 dB <sub>20</sub>	-0.2 rad = -11°
5	0.8 = -1.9 dB <sub>20</sub>	-0.9 rad = -52°
10	0.5 = -6 dB <sub>20</sub>	-1.6 rad = -92°
20	0.2 = -14 dB <sub>20</sub>	-2.2 rad = -126°

c) See Figure 4.8a in Solutions.

4.9  $G_1$ -B,  $G_2$ -D,  $G_3$ -A,  $G_4$ -C,  $G_5$ -E.

4.10 Bode gain-step response pairs: A-D, B-C, C-A, D-B.

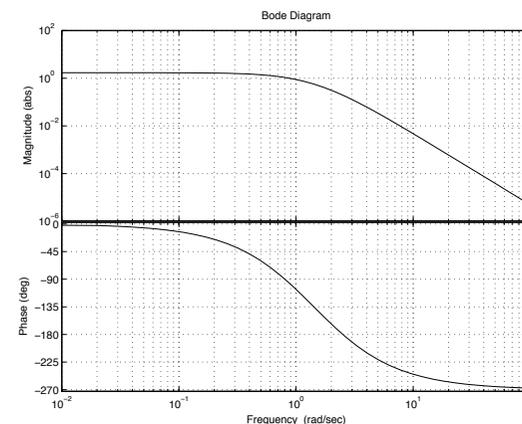


Figure 4.11a

4.11 a) The bode digram of the system is shown in Figure 4.11a.

b)  $K = \frac{1}{0.1946} = 5.14$

4.12 a)  $K \leq 5.04$

b)  $F(s) = 1.58 \frac{100s+1}{100s}$

4.13 a) B - a - 1 - II and A - b - 2 - I.

b) B - b - 1 - I and A - a - 2 - II.

# 5 Compensation

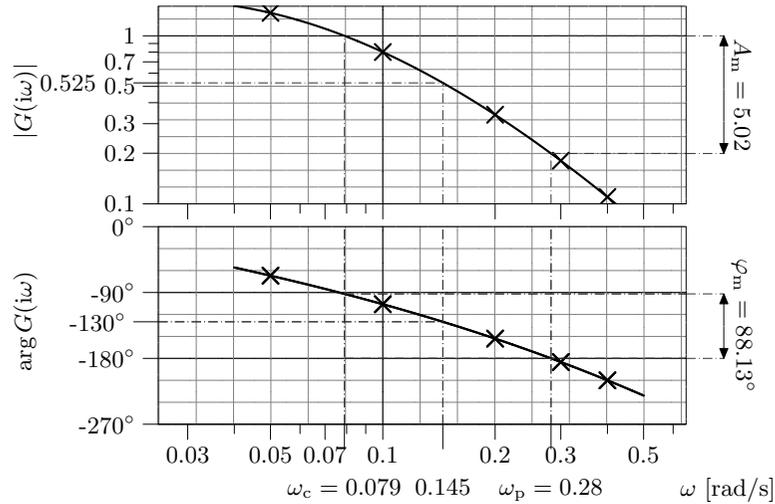


Figure 5.2a

5.1 For example, the following controller fulfills the requirements:

$$F(s) = 0.96 \frac{6.25s + 1}{1.56s + 1} \frac{(31.25s + 1)}{(31.25s + 0.1)}$$

5.2 a) See Figure 5.2a.

b) Largest crossover frequency: 0.14 rad/s.

c) One controller that fulfills the requirements is the lead compensator (with gain adjustment)

$$F(s) = 1.9 \cdot 7 \frac{s + 0.106}{s + 0.106 \cdot 7}$$

5.3 a) See Figure 5.3a.

b) Smallest value of ramp error 0.067 and crossover frequency 150 rad/s.

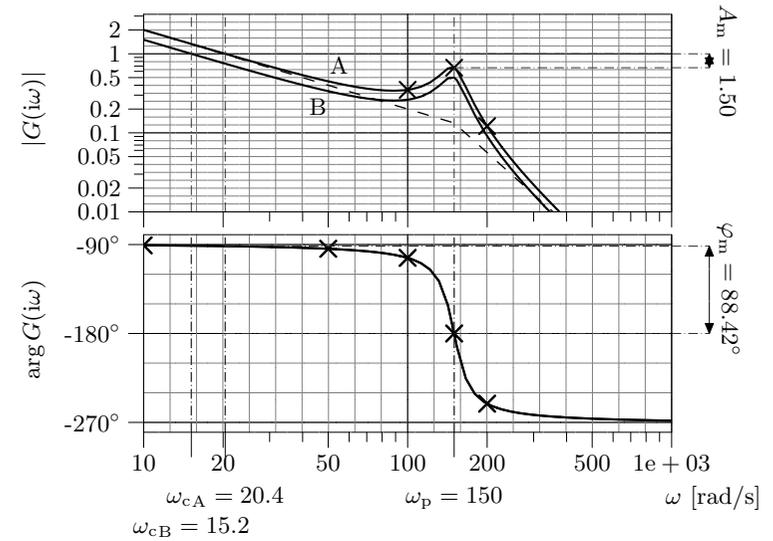


Figure 5.3a

c) One controller that fulfills the constraints is

$$F(s) = 0.75 \frac{s + 1.4}{s + 0.1}$$

5.4 One controller which fulfills the requirements is

$$F(s) = 1.12 \cdot 4 \cdot \frac{s + 6.7}{s + 27} \cdot \frac{s + 1.33}{s + 0.15}$$

5.5 A-E-C, B-C-E, C-A-B, D-D-D, E-B-A.

5.6 One controller which satisfies the demands is

$$F(s) = 1.2 \cdot 5 \frac{s + 8.0}{s + 5 \cdot 8.0} \cdot \frac{s + 1.8}{s + 1.8/84}$$

5.7 The system is stable when  $0 < K < 0.2$  or  $1.67 < K < 5$ .

5.8 a)  $T < 0.698$  s

b)  $0.1 \text{ s} < T < 0.279$  s

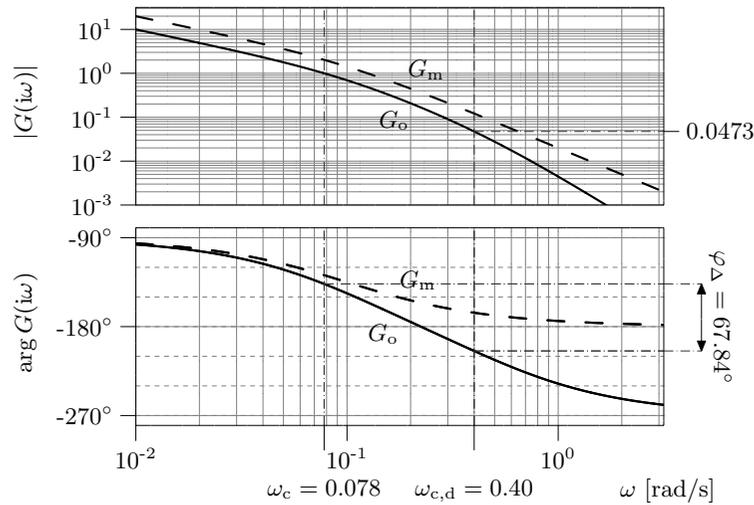


Figure 5.9a

5.9 a)  $k_A = 0.25$  and  $a = 0.5$ . The Bode plot is given in Figure 5.9a.

b) One controller that does the job is

$$F(s) = 10.6 \cdot \left( 4 \frac{(s+0.2)}{(s+0.2 \cdot 4)} \right)^2$$

5.10 The following compensator fulfills the requirements:

$$F(s) = 4.4 \cdot \left( 4 \frac{s+0.53}{s+0.53 \cdot 4} \right)^2 \frac{s+0.105}{s+0.105/195}$$

5.11 a) See Figure 5.11a.

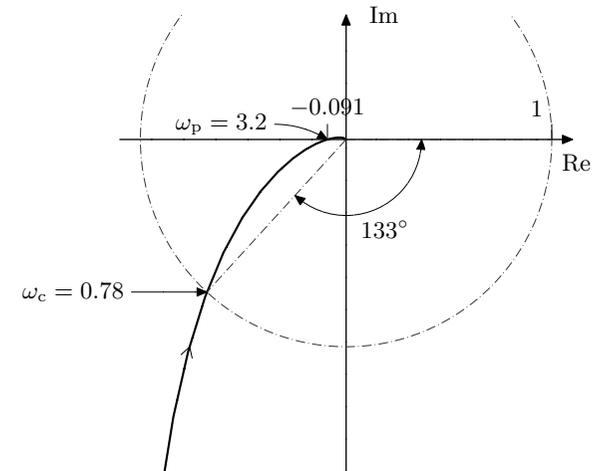


Figure 5.11a

b) Asymptotically stable for  $0 < K < 10$ .

c)  $\lim_{t \rightarrow \infty} e(t) = 5$

d)  $T < 0.4$

5.12 a) Impossible to determine.

b) It is stable.

5.13 a)  $\omega_c = 5$  rad/s,  $\omega_p = 9.5$  rad/s,  $A_m = 3.5$  and  $\varphi_m = 27^\circ$ .

b)-d) See solution.

5.14 The following controller will do:

$$F(s) = 39 \frac{7.9s+1}{5s+1} \frac{50s+1}{50s}$$

5.15 a)  $e_0 = 0$ ,  $e_1 = \frac{4}{K}$ , provided  $K < 4000$ . Larger  $K$  results in an unstable system.

b) The following controller will do:

$$F(s) = 721 \frac{0.0277s + 1}{0.0036s + 1} \cdot \frac{0.1s + 1}{0.1s + 0.18}$$

5.16 a) The maximum cross-over frequency is  $\omega_c = 0.27$  rad/s and it is achieved for  $K = 2.86$ .

b) The steady state error is approximately 1.75.

c)

$$F(s) = 2.39 \frac{(4.4s + 1)}{(0.7 \cdot 4.4s + 1)} \frac{(37s + 1)}{(37s + 0.1)}$$

5.17 The combinations are: A - III, B - I, C - II, and D - IV.

5.18 a) One possible solution is  $p = 1$ ,  $n = 2$ , and  $m = 0$ .

b)

$$F(s) = 3.6 \frac{(0.92s + 1)}{(0.92 \cdot 0.13s + 1)} \frac{(3.3s + 1)}{(3.3s + 0.036)}$$

5.19 A - *iii*, B - *ii*, C - *i* and D - *iv*

5.20 a)  $G_c(s) = \frac{F(s)G(s)}{1+F(s)G(s)}$

b)  $\omega_c = 1$  rad/s,  $\varphi_m = 50^\circ$ ; yes, because  $\varphi_m > 0^\circ$ .

c)  $K = 3.33$

d) The phase margin has decreased, which in the time domain means that the overshoot increased for the closed-loop system.

e)  $\beta = 0.18$ ,  $\tau_D = 1.18$ s,  $K = 1.4$

f)  $e(t \rightarrow \infty) = 0.17$ ; to decrease the error,  $F(0)$  has to increase.

g)  $\gamma = 0$ ,  $F(0) \rightarrow \infty$

h)  $\gamma = 0.048$ ,  $\tau_I = 5$ s. If  $\tau_I$  is too small, the phase margin decreases more; if  $\tau_I$  is too large, the steady state value is only reached in practice at very small frequencies.

## 6 Sensitivity and Robustness

6.1 The gain of the sensitivity is:

$$|S(i\omega)| = \frac{\sqrt{2}}{\sqrt{(K-1)^2 + 1}}$$

and the requirement on  $K$  becomes  $K > 2$ .

6.2 The maximum bandwidth is  $\omega_B = 1$ .

6.3 See the solution, Figure 6.3a.

6.4 Yes.

6.5 a) See the solution, Figure 6.5a.

b)

$$\left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right| < \frac{1}{2}$$

6.6 a) No, stability cannot be guaranteed when  $\tilde{G}(s) = 1$ .

b)  $\alpha > -2/25$ . This is not contradictory since the robustness criterion is a *sufficient* but not *necessary* condition.

6.7 a) Asymptotically stable for  $\alpha > 3$ . See the solution, Figure 6.7a.

b)  $\alpha > 4$

c) The robustness criterion gives a sufficient but not necessary condition.

6.8  $0 \leq \gamma < \frac{1}{35}$

6.9  $y(t) = \frac{1}{\sqrt{2}} \sin(t - \frac{\pi}{4}) - \sin(t)$

6.10 a)  $\frac{1}{G_\Delta(s)} = -\frac{s+1}{s}$

b) Stability cannot be guaranteed for  $F(s) = 1$ , while it can be guaranteed for the regulator from Problem 5.13.

6.11 The amplitude of the steady state error will be 0.2.

6.12 The controller also stabilizes the system for the stirring speed 400 r/min.

6.13  $K \geq \sqrt{396} \approx 19.9$

## 7 Special Controller Structures

- 7.1 a) See Figure 7.1a in the solution.  $\omega_c$  and  $\varphi_m$  are undefined and  $A_m = 43.5$ .  
The stability requirement gives  $K_1 = 21.75$  which implies

$$\lim_{t \rightarrow \infty} e(t) = 0.0487 \cdot a$$

where  $a$  is the size of the step.

- b) See Figure 7.1b in the solution.  $\omega_c$  and  $\varphi_m$  are undefined, and  $A_m = 16$ .  
The requirement gives  $K_1 = 8$  which implies

$$\lim_{t \rightarrow \infty} e(t) = 0.111 \cdot a$$

- 7.2 a)  $F_f(s) = 1$ , and  $h(t) = -\frac{0.1}{A \cdot 2}(1 - e^{-2t})$ .

- b) Zero steady state error.

- 7.3 a)

$$F_f = -\frac{G_v}{G_u} = -\frac{3(s+3)}{2(s+4)}$$

- b) The amplitude of the control signal is 3.

c)  $\lim_{t \rightarrow \infty} y(t) = \frac{9(1-b/2)}{12+4Kb}$

- 7.4 a)  $F_f(s) = -\frac{4(s+1)}{3(s+2)(s+5)}$

b)  $\lim_{t \rightarrow \infty} y(t) = -0.012$

c)  $\lim_{t \rightarrow \infty} y(t) = -\frac{0.012}{3K+1}$

- d)  $y(t)$  doesn't have a final value.

# 8 State Space Description

8.1

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & -1/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ K \end{pmatrix} u \\ y &= (1 \quad 0) x\end{aligned}$$

8.2

$$\begin{aligned}\dot{x}_{1\Delta} &= x_{2\Delta} \\ \dot{x}_{2\Delta} &= \omega_0^2 x_{1\Delta} + u_\Delta \\ y_\Delta &= x_{1\Delta}\end{aligned}$$

8.3

$$\begin{aligned}\dot{x}_1(t) &= K_2 x_2(t) + M_1(t) \\ \dot{x}_2(t) &= -x_1(t) + x_3(t) \\ \dot{x}_3(t) &= -K_2 x_2(t) + K_1 i(t)\end{aligned}$$

8.4 a)

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= -6x_1(t) - 11x_2(t) - 6x_3(t) + 6u(t) \\ y(t) &= x_1(t)\end{aligned}$$

b)

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + x_3(t) + 4u(t) \\ \dot{x}_2(t) &= -3x_1(t) + 2u(t) \\ \dot{x}_3(t) &= -5x_1(t) + x_2(t) + u(t) \\ y(t) &= x_1(t)\end{aligned}$$

c)

$$\begin{aligned}\dot{x}_1(t) &= -2x_1(t) - u(t) \\ \dot{x}_2(t) &= -3x_2(t) + 3u(t) \\ y(t) &= x_1(t) + x_2(t)\end{aligned}$$

8.5

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + 2u(t) \\ \dot{x}_2(t) &= -4x_2(t) + 3u(t) \\ y(t) &= x_1(t) + x_2(t)\end{aligned}$$

8.6  $G(s) = \frac{s}{(s+2)(s+3)}$

8.7

$$x(t_0 + T) = e^{AT} x(t_0) + \left( \int_{t_0}^{t_0+T} e^{A(t_0+T-s)} ds \right) Bu_0$$

8.8 a) The state space description of the closed loop system

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -3 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} h_{\text{ref}}(t) \\ h(t) &= (1 \quad 0 \quad 0) x(t)\end{aligned}$$

b,c) The state space description of the closed loop system with noise

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -3 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} n(t) \\ h(t) &= (1 \quad 0 \quad 0) x(t)\end{aligned}$$

8.9 Yes, since the system is controllable.

8.10 a) Dimensions: 2 and 1. Subspaces:  $\left\{ \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}^T, \begin{pmatrix} -2 & 3 & -6 \end{pmatrix}^T \right\}$  and  $\left\{ \begin{pmatrix} 0 & -1 & 2 \end{pmatrix}^T \right\}$ .

b) Dimensions: 2 and 1. Subspaces:  $\left\{ \begin{pmatrix} 0 & 4 & -2 \end{pmatrix}^T, \begin{pmatrix} 0 & -8 & 8 \end{pmatrix}^T \right\}$  and  $\left\{ \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T \right\}$ .

8.11 a)  $x_1 = 1 - e^{-t}$ ,  $x_2 = 0.5(e^{2t} - 1)$

b) No. Yes.

c) Controllable, not observable.

d) Unobservable growing state  $\Rightarrow$  simulation collapses.

8.12 Poles:  $1 \pm i\sqrt{2}$ . Zeros:  $-1$ .

8.13 a)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{\alpha}x_1 - \frac{u}{\alpha} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= x_3 - u\end{aligned}$$

b)  $\det \mathcal{S} = \frac{1}{\alpha^2}(1 - \frac{1}{\alpha})^2$ . Thus, the system is controllable except for the case  $\alpha = 1$ , that is, when the two pendulums have the same lengths.

8.14 a)

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -1 & 0 \\ 1 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \\ y &= (1 \quad 1) x\end{aligned}$$

b)  $u = -5x_1 + x_2 + 3.2r$

c)  $Y(s) = \frac{3.2(2s+5)}{(s+4)^2}R(s)$

8.15 a) The model is given by

$$\begin{aligned}V \frac{dc_A}{dt} &= -Vk_1c_A^3 + qc_{A,in} - qc_A \\ V \frac{dc_B}{dt} &= \frac{Vk_1c_A^3}{3} - qc_B\end{aligned}$$

b) The linearized model is given by

$$\begin{aligned}\frac{d}{dt} \begin{pmatrix} c_{A,\Delta} \\ c_{B,\Delta} \end{pmatrix} &= \begin{pmatrix} \frac{-q-3k_1c_A^*V}{V} & 0 \\ k_1c_A^* & \frac{-q}{V} \end{pmatrix} \begin{pmatrix} c_{A,\Delta} \\ c_{B,\Delta} \end{pmatrix} + \begin{pmatrix} \frac{q}{V} \\ 0 \end{pmatrix} u \\ y &= (0 \quad 1) \begin{pmatrix} c_{A,\Delta} \\ c_{B,\Delta} \end{pmatrix}\end{aligned}$$

8.16 a)

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -k_{21} & k_{12} \\ k_{21} & -k_{12} \end{pmatrix} x + \begin{pmatrix} 0 \\ K \end{pmatrix} u \\ y &= (0 \quad 1) x\end{aligned}$$

with

$$x = (x_1 \quad x_2)^T$$

b) The poles are  $s = 0$  and  $s = -(k_{12} + k_{21})$ .

c) The system is observable when  $k_{21} \neq 0$ . If  $k_{21} = 0$  we are not able to observe state  $x_1$  by measuring  $x_2$ .

d) The nonlinear state space is given by

$$\begin{aligned}\dot{x}_1 &= -k_{21}x_1 + \frac{V_{\max}x_2}{K_M + x_2} \\ \dot{x}_2 &= -\frac{V_{\max}x_2}{K_M + x_2} + k_{21}x_1 + Ku\end{aligned}$$

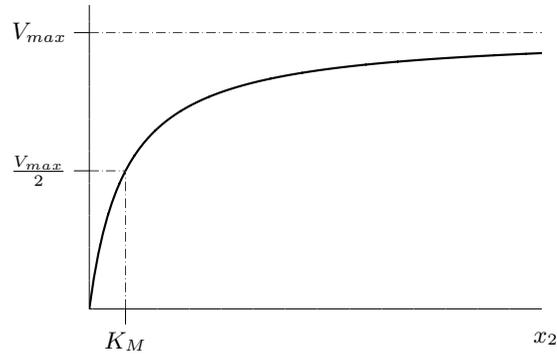


Figure 8.16a

e) The relation is shown in Figure 8.16a.

f) Linearization gives

$$A = J_f(x_0, u_0) = \begin{pmatrix} -k_{21} & \frac{V_{\max}}{4K_M} \\ k_{21} & -\frac{V_{\max}}{4K_M} \end{pmatrix}.$$

and

$$B = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 \\ K \end{pmatrix}.$$

With  $x_\Delta = x - x_0$  and  $u_\Delta = u - u_0$ , the linear approximation of the system around  $x_0, u_0$  is given by

$$\dot{x}_\Delta = Ax_\Delta + Bu_\Delta.$$

g)

$$\dot{x} = \begin{pmatrix} -k_{21} & k_{12} & 0 \\ k_{21} & -(k_{12} + k_{32}) & k_{23} \\ 0 & k_{32} & -k_{23} \end{pmatrix} x + \begin{pmatrix} 0 \\ K \\ 0 \end{pmatrix} u$$

$$y = (0 \quad 1 \quad 0) x$$

h) The determinant of  $\mathcal{O}$  can be expanded along the first row to obtain

$$\det \mathcal{O} = k_{21}k_{23}(k_{23} - k_{21}).$$

I.e. the system is observable unless  $k_{21} = 0$  or  $k_{23} = 0$  or  $k_{23} = k_{21}$ . The first two cases mean that the corresponding state,  $x_1$  and  $x_3$  respectively, do not influence  $x_2$  and would then not be visible in the output. In the third case,  $k_{23} = k_{21}$ ,  $x_1$  and  $x_3$  do indeed influence  $x_2$  but they do it in the exact same way and we can therefore not distinguish between them.

8.17 a) The stationary point is

$$x_0 = \begin{pmatrix} \frac{k_1(k_{12}+k_2)}{k_{21}k_2} & \frac{k_1}{k_2} \end{pmatrix}^T.$$

b)

$$G(s) = \frac{k_1k_{21}}{(s + k_{21})(s + k_{12} + k_2) - k_{12}k_{21}}.$$

c)

$$\lim_{t \rightarrow \infty} y(t) = \frac{k_1}{k_2}.$$

Since the output is defined as  $y = x_2$ , it approaches the stationary point  $x_{0,2}$ .

8.18 b)

$$A = \begin{pmatrix} 0 & 1 \\ \frac{2g}{x_{10}} & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ -\frac{k}{mx_{10}^2} \end{pmatrix}$$

$$C = (1 \quad 0)$$

## 9 State Feedback

- 9.1 a) State feedback. Poles in  $\{-3, -5\}$  gives the state feedback

$$u = -6x_1 - 14x_2 + y_{\text{ref}}$$

Poles in  $\{-10, -15\}$  gives the state feedback

$$u = -23x_1 - 149x_2 + y_{\text{ref}}$$

- b) Observer poles in  $-20$  gives the observer gain

$$K = \begin{pmatrix} 38 \\ -399 \end{pmatrix}$$

- 9.2 a)

$$\dot{x} = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ K_1 \end{pmatrix} u$$

b)  $u = -\frac{1}{8K_1K_2}x_1 - \frac{3}{4K_1}x_2 - \frac{3}{2K_1}x_3$

c) Observer gain  $K^T = (6 \quad 12/K_2 \quad 8/K_2)$

9.3 a)  $u = -\frac{2}{c_1\tau^2}\theta - \frac{1}{\tau c_1}\omega + \frac{2}{c_1\tau^2}\theta_{\text{ref}}$

b)  $u = -\frac{2}{c_1\tau^2}\theta - \frac{1}{\tau c_1}\omega + \frac{2}{c_1\tau^2}\theta_{\text{ref}} - \frac{c_2}{c_1}\hat{x}_3$

9.4 State feedback gain  $L = (6 \quad -2)$ . Observer gain  $K^T = (16 \quad 9)$ .

9.5 The system is observable and the poles of the observer may be placed arbitrarily.

- 9.6 a) Yes, since the system is controllable.

- b) Closed loop poles in  $-3$  gives

$$u = -3x_1 - 5x_2 - 4x_3 + y_{\text{ref}}$$

- c) The system is observable with the sensor at  $x_1$  or  $x_3$ . The sensor at  $x_1$  and observer poles in  $-4$  give  $K^T = (6 \quad 14 \quad 14)$ .

9.7  $\hat{X}_3(s) = \frac{K_1}{s+K}U(s) + \frac{K^2s}{s+K}X_2(s)$

9.8 a)  $L = (1 \quad 2)$

- b) In steady state:  $h = -0.1$ .

- c)  $F_f(s) = 2$  gives, in steady state,  $h = 0$ .

d)  $h = \frac{2(k_1-1)}{k_1}v$

- e) Introduce the integral of the height as a new state

$$z(t) = \int_0^t h(s) ds \Rightarrow \dot{z} = h$$

9.9  $T < \frac{\arctan 2\omega_c}{\omega_c} = 0.65s$

9.10 a)  $K^T = (-13 \quad 38)$

- b) The transfer function from  $v$  to  $\tilde{x}_1$  is

$$-C_1(sI - A + KC)^{-1}K = \frac{13s - 12}{s^2 + 15s + 50}$$

where  $C_1 = (1 \quad 0)$ .

- 9.11 a) The initial value theorem gives

$$\dot{y}(0) = -\frac{\beta^2}{\alpha}$$

and hence  $\dot{y}(0)$  decreases as  $\alpha$  decreases.

- b) No, since the zero is not affected by the feedback.

9.12 A very fast closed loop system:

- implies that the poles are far into the LHP which implies a need for generating large input signals.
- easily becomes unstable in case of model uncertainties.
- becomes sensitive to measurement noise.
- has a sensitivity function with a large peak.

9.13 a)  $L = \begin{pmatrix} 1 & 0 \end{pmatrix}$

b)  $r(t) = r_0 e^{-5t}$

9.14 a)  $x_2 = y$  (motor angle) and  $x_1 = \dot{y}$  (angular velocity).

b) The pole locations give similar rise and settling times. With complex poles the maximum value of the input is lower.

c) Larger weight on the motor angle gives faster response.

d) Increasing weight on the input makes the system slower.

e) Increasing weight on the velocity makes the system slower.

9.15 a) Yes the system is controllable.

b) Poles in  $-0.1$  gives the state feedback

$$u(t) = -0.13x_1(t) - 0.128x_2(t)$$

c) It is desirable that the estimation error converges to zero faster than the dynamics of the system. Thus, we should place the eigenvalues of the observer to the left of the poles of the closed loop system. To avoid large amplification of the measurement noise the poles of the observer should not be placed too far into the left hand plane.

d) Observer poles in  $-0.1$  gives the observer gain

$$K = \begin{pmatrix} 0.45 \\ 0.33 \end{pmatrix}$$

9.16 Are the specifications 1–4 fulfilled?

1. The bandwidth requirement is not fulfilled.
2. The system is stable despite the model errors.
3. The gain is different from 1 when  $\kappa \neq 1$
4. Both measurement and process noise are amplified for some frequency.

9.17 a)  $u = 4x_1 + x_2$

b) Yes! It is essential that the input  $u$  is known since  $u$  is required in the observer design to get an asymptotically vanishing state estimation error.

c) Yes, by introducing a third state  $x_3 = u$ . This new system is observable hence a observer can be designed to estimate  $u$ .

9.18 a) The poles are pure complex and thus the system doesn't have a well defined stationary error or speed of response.

b) A linear combination of  $r$  and  $x_2$  is given by

$$u = l_0 r - l_2 x_2$$

with this controller the poles can be placed with  $l_2$  as

$$s = \frac{-l_2}{2} \pm \sqrt{\frac{l_2^2 - 4}{4}}$$

and by setting  $l_0 = 1$  the stationary error will be zero when  $w = 0$ . If  $w \neq 0$  and  $l_0 = 1$  then there will be stationary error of size  $l_2 w$ .

c) Designing a observer with the following observer gains  $k_1 = -11$ ,  $k_2 = 6$ , and  $k_3 = -8$ . Let the control law be  $u = l_0 r - l_2 \hat{x}_2 - l_3 \hat{x}_3$ . With  $l_3 = l_2$  and  $l_0 = 1$  there will be no error. Place the poles to the closed loop with  $l_2$ .

# 11 Implementation

11.1  $\beta_1 = 0.905$ ,  $\alpha_1 = 19.14$ , and  $\alpha_2 = -18.95$ .

11.2 a)  $y_{k+1} - y_k = Tu_k$

b)  $0 < K < \frac{2}{T}$

11.3 a)

$$A = \frac{1}{\sqrt{1 + (\omega_2 T_1)^2}}$$
$$\omega_1 = \frac{2\pi}{T} - \omega_2$$
$$\varphi = \pi + \arctan \omega_2 T_1$$

b)  $T_1 = T/\pi$  gives  $A = \frac{1}{\sqrt{1 + (\omega_2 T/\pi)^2}}$ .

# Liten reglerteknisk ordlista

This version: August 16, 2016



# 1 Engelsk-svensk

actuator	ställdon
amplitude	amplitud
attenuation	dämpning
bandwidth	bandbredd
bond graph	bindningsgraf
closed loop system	slutet system
control law	styrslag
controllability	styrbarhet
controller	regulator
convolution	faltning
correlation analysis	korrelationsanalys
credibility	trovärdighet
crossover frequency	skärfrekvens
damping	dämpning
damping ratio	relativ dämpning
describing function	beskrivande funktion
discrete event systems	händelseorienterade system
distributed parameter models	fördelade parametriska modeller
disturbance rejection	störningsundertryckning
eigenvalue	egenvärde
feedback	återkoppling
feedforward	framkoppling
flow	flöde
gain	förstärkning
gain crossover frequency	(amplitud)skärfrekvens
gain margin	amplitudmarginal
impulse response	impulssvar
initial value	begynnelsevärde
loop gain	kretsförstärkning, öppna systemet
lumped models	aggregerade modeller
magnitude	amplitud
observability	observerbarhet
observer	observatör
open-loop system	öppet system, kretsförstärkning

overfit	överanpassning
overshoot	översläng
parsimonious	sparsam
peak frequency	resonansfrekvens
peak resonance	resonanstopp
phase crossover frequency	fasskärfrekvens
phase lag	fasretarderande
phase lead	fasavancerande
phase margin	fasmarginal
ramp function	ramp
rank	rang
reset windup	integratoruppvridning
resonant frequency	resonansfrekvens
rise time	stigtid
root locus (pl. loci)	rotort
sensitivity function	känslighetsfunktion
sensor	givare
settling time	insvängningstid, lösningstid
sinusoidal	sinusformad
stability robustness	stabilitetsrobusthet
state	tillstånd
state feedback	tillståndsåterkoppling
static gain	statisk förstärkning
steady state	stationärt tillstånd
step function	steg
step response	stegsvar
subspace	underrum
time delay	tidsfördröjning
transfer function	överföringsfunktion
unit step	enhetsteg
unstable	instabil
validity	giltighet
whitening filter	blekningsfilter

## 2 Svensk-engelsk

(amplitud)skärfrekvens	gain crossover frequency	observatör	observer
aggregerade modeller	lumped models	observerbarhet	observability
amplitud	amplitude	ramp	ramp function
amplitud	magnitude	rang	rank
amplitudmarginal	gain margin	regulator	controller
bandbredd	bandwidth	relativ dämpning	damping ratio
begynnelsevärde	initial value	resonansfrekvens	peak frequency
beskrivande funktion	describing function	resonansfrekvens	resonant frequency
bindningsgraf	bond graph	resonanstopp	peak resonance
blekningsfilter	whitening filter	rotort	root locus (pl. loci)
dämpning	damping, attenuation	sinusformad	sinusoidal
egenvärde	eigenvalue	slutet system	closed loop system
enhetsteg	unit step	skärfrekvens	gain crossover frequency
faltning	convolution	sparsam	parsimonious
fasavancerande	phase lead	stabilitetsrobusthet	stability robustness
fasmarginal	phase margin	stationärt tillstånd	steady state
fasretarderande	phase lag	statisk förstärkning	static gain
fasskärfrekvens	phase crossover frequency	steg	step function
flöde	flow	stegsvar	step response
framkoppling	feedforward	stigtid	rise time
fördelade parametriska modeller	distributed parameter models	styrbarhet	controllability
förstärkning	gain	styrslag	control law
giltighet	validity	ställdon	actuator
givare	sensor	störningsundertryckning	disturbance rejection
händelseorienterade system	discrete event systems	tidsfördröjning	time delay
impulssvar	impulse response	tillstånd	state
instabil	unstable	tillståndsåterkoppling	state feedback
insvängningstid, lösningstid	settling time	trovärdighet	credibility
integratoruppvridning	reset windup	underrum	subspace
korrelationsanalys	correlation analysis	återkoppling	feedback
kretsförstärkning	loop gain, open loop system	öppet system	open-loop system, loop gain
känslighetsfunktion	sensitivity function	överanpassning	overfit
		överföringsfunktion	transfer function
		översläng	overshoot

# Introduktion till Control System Toolbox 5.0

This version: August 16, 2016



# 1 Inledning

Denna skrift är en kort inledning till hur MATLAB och Control System Toolbox (CST) används i kurserna i Reglerteknik.

# 2 System

I Control System Toolbox finns datastrukturer för att hantera så kallade *LTI-objects*, dvs linjära tidsinvarianta system, på ett bekvämt sätt. Vi kommer inledningsvis främst att arbeta med system på överföringsfunktionsform, men senare även med system på tillståndsform. Ett objekt som representerar ett system på överföringsfunktionsform skapas med funktionen `tf`. Detta kan göras på två olika sätt, och det första alternativet visas i exemplet nedan.

Betrakta överföringsfunktionen

$$G(s) = \frac{4}{s(s^2 + 2s + 4)}$$

Mata in systemet och ge objektet namnet `G`. Argumenten till funktionen `tf` utgörs av radvektorer innehållande täljarens respektive nämnarens koefficienter.

```
>> G = tf( 4, [ 1 2 4 0 ] )  
  
Transfer function:  
4  
-----  
s^3 + 2 s^2 + 4 s
```

Med det andra alternativet kan man mata in överföringsfunktionen på symbolisk form genom att först skapa ett objekt bestående av symbolen `s`. Därefter kan man `ex` addera och multiplicera med denna symbol på samma sätt som görs med Laplace-variabeln `s` vid handräkning.

Skapa ett objekt bestående av symbolen `s`. Bilda överföringsfunktionen genom att använda vanliga räkneoperationer.

```
>> s = tf( 's' );  
>> G = 4 / ( s * ( s^2 + 2*s + 4 ) )  
  
Transfer function:  
4  
-----  
s^3 + 2 s^2 + 4 s
```

En finess med överföringsfunktioner representerade som LTI-objekt är att man kan multiplicera och addera överföringsfunktioner på ett rättfram sätt.

Skapa en ny överföringsfunktion `G2` genom att seriekoppla `G(s)` och överföringsfunktionen

```
>> G2 = G * 1 / ( s + 1 )  
  
Transfer function:  
4  
-----  
s^4 + 3 s^3 + 6 s^2 + 4 s
```

$$\frac{1}{s + 1}$$

# 3 Poler och nollställen

Poler och nollställen till överföringsfunktioner beräknas med funktionerna `pole` respektive `tzero`. Poler och nollställen kan även ritas med funktionen `pzmap`.

Beräkna polerna till `G(s)`. Systemet har en reell pol i origo och två komplexa poler.

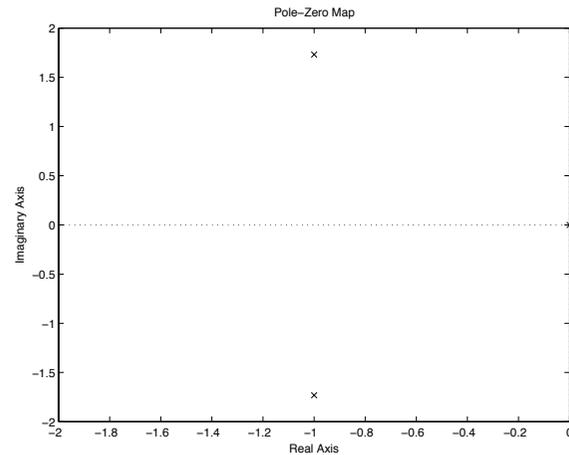
```
>> pole( G )  
  
ans =  
  
0  
-1.0000 + 1.7321i  
-1.0000 - 1.7321i
```

Beräkna nollställena till `G(s)`. Eftersom täljaren i överföringsfunktionen är konstant saknar systemet nollställen.

```
>> tzero( G )  
  
ans =  
  
Empty matrix: 0-by-1
```

Rita in systemets poler och nollställen i det komplexa talplanet. Poler markeras med kryss och nollställen, i de fall de förekommer, markeras med ringar.

```
>> pzmap( G )
>> axis([ -2 0 -2 2 ])
```



## 4 Återkoppling

I kursen behandlas återkopplade reglersystem enligt figur 1.

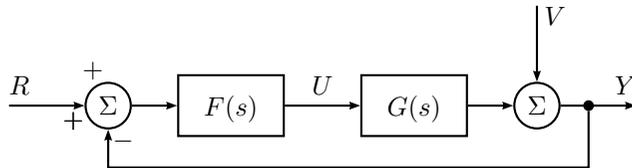


Figure 1. Reglersystem

Med systembeskrivningen

$$Y(s) = G(s)U(s) + V(s)$$

och återkopplingen

$$U(s) = F(s)(R(s) - Y(s))$$

ges det återkopplade systemet av

$$Y(s) = G_c(s)R(s) + S(s)V(s)$$

där

$$G_c(s) = \frac{F(s)G(s)}{1 + F(s)G(s)}$$

och

$$S(s) = \frac{1}{1 + F(s)G(s)}$$

Överföringsfunktionerna för det återkopplade systemet kan beräknas med funktionen **feedback**.

Generera överföringsfunktionen för en proportionell regulator med förstärkning  $K_p = 0.7$ .

```
>> F = tf( 0.7 )
```

Transfer function:  
0.7

Beräkna överföringsfunktionen för det återkopplade systemet.

```
>> Gc = feedback( F * G, 1 )
```

Transfer function:  
2.8

$$s^3 + 2 s^2 + 4 s + 2.8$$

Beräkna känslighetsfunktionen.

```
>> S = 1 / ( 1 + F * G )
```

Transfer function:  
 $s^3 + 2 s^2 + 4 s$

$$s^3 + 2 s^2 + 4 s + 2.8$$

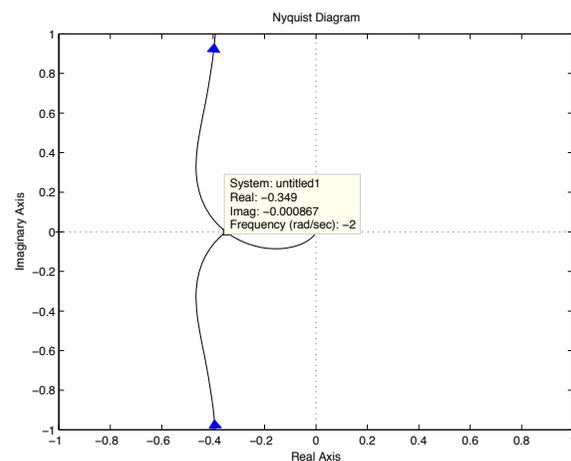
I exemplet ovan hade vi kunnat beräkna  $G_c$  på motsvarande sätt som  $S$  beräknades, dvs  $G_c = F * G / (1 + F * G)$ . Med denna metod får dock täljaren och nämnaren i  $G_c$  ett antal gemensamma faktorer som kan förkortas bort. Genom att använda funktionen **feedback** undviks detta. De gemensamma faktorerna i det första alternativet kan elimineras genom att använda funktionen **minreal(Gc)** Testa själv och jämför.

## 5 Nyquistdiagram

Nyquistkurvor för en eller flera överföringsfunktioner ritas med funktionen `nyquist`. Eftersom funktionen `nyquist` graderar axlarna automatiskt kan diagrammet ibland bli svårläst. Läsbarheten kan förbättras genom att man själv väljer axlarnas gradering med funktionen `axis`. Man kan få ut mycket information ur figuren genom att använda vänster respektive höger musknapp. Med vänster musknapp kan man t ex markera en punkt på kurvan och få ut motsvarande värde på  $\omega$  samt nyquistkurvans värde i denna frekvens. Med höger musknapp får man en meny med olika operationer som kan göras med figuren.

Rita nyquistkurvan för det öppna systemet då systemet  $G(s)$  styrs med en proportionell återkoppling med förstärkning  $K_P = 0.7$ . Justera axlarnas gradering och markera punkten där nyquistkurvan passerar negativa delen av reella axeln.

```
>> nyquist( F * G )
>> axis([ -1 1 -1 1 ])
```



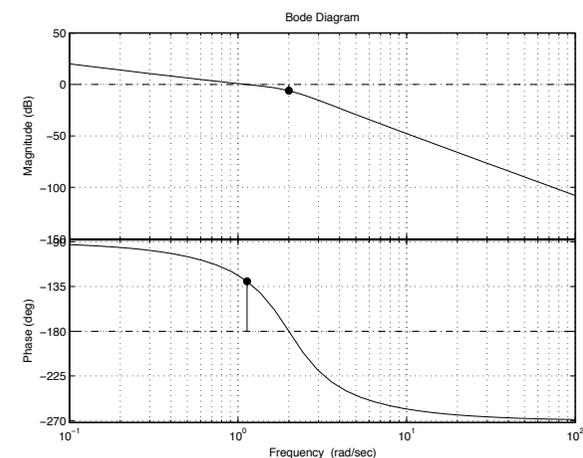
## 6 Bodediagram

Bodediagram för en eller flera överföringsfunktioner ritas med funktionen `bode`. Även i detta fall kan man läsa av punkter i figuren genom att markera med vänster

musknapp. Med höger knapp får man en meny där man t ex kan välja att markera frekvenserna där stabilitetsmarginalerna läses av.

Beräkna frekvensfunktionen för systemet  $G$  och rita upp den i ett bodediagram. Notera att amplitudkurvan graderas i decibel. Använd höger musknapp och lägg in rutnät i figuren samt markera var fas- och amplitudskärfrekvenserna ligger.

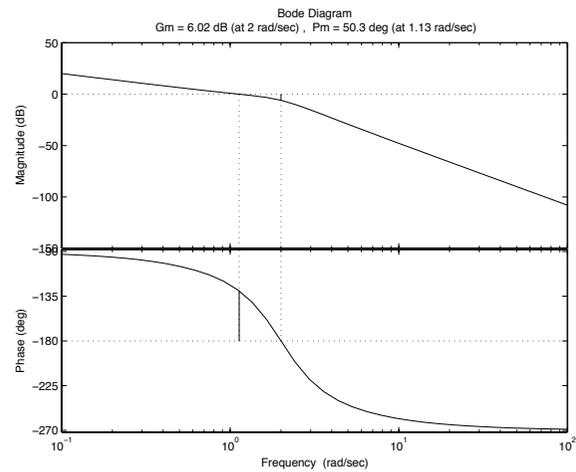
```
>> bode( G )
```



För att bestämma skärfrekvenser samt fas- och amplitudmarginal kan man även använda funktionen `margin`, vilken förutom att rita upp amplitud- och faskurvorna även skriver ut dessa värden.  $G_m$  och  $P_m$  betecknar amplitud- respektive fasmarginal.

Beräkna frekvensfunktionen för systemet  $G$  och rita upp den i ett bodediagram.

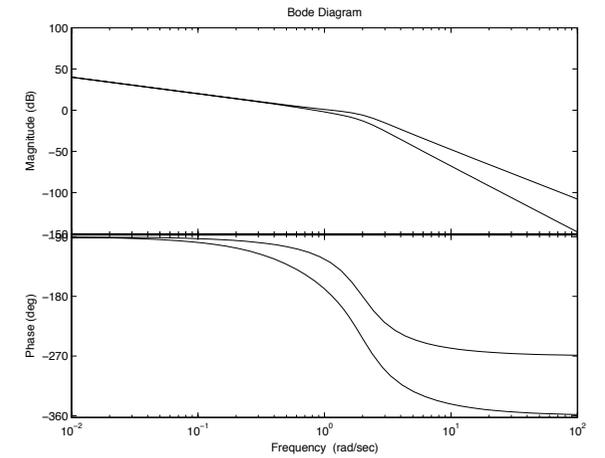
```
>> margin( G )
```



För att t ex kunna göra jämförelser mellan två frekvensfunktioner kan dessa ritas i samma diagram.

Beräkna frekvensfunktionerna för systemen  $G$  och  $G2$ .

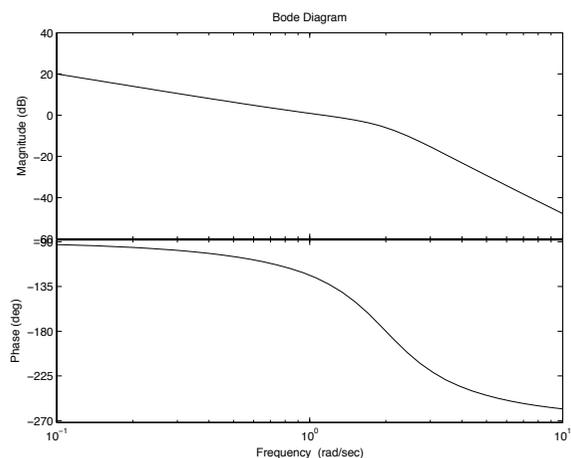
```
>> bode( G, G2 )
```



Skalan på frekvensaxeln kan väljas genom att som sista argument i funktionsanropet ange största och minsta frekvensvärdet mellan krullparenteser .

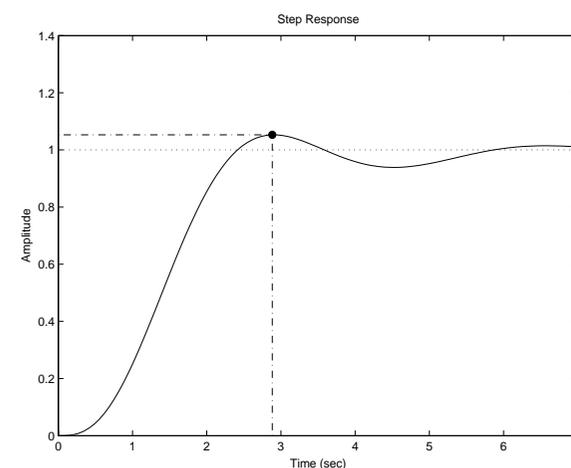
Beräkna frekvensfunktionen för systemet  $G$  från 0.1 till 10 rad/s och rita upp den i ett bodediagram.

```
>> bode( G, { 0.1, 10 } )
```



Beräkna och rita upp det återkopplade systemets stegsvar. Markera stegsvarets översläng.

```
>> step( Gc )
```



## 7 Simulering

### 7.1 Stegsvar

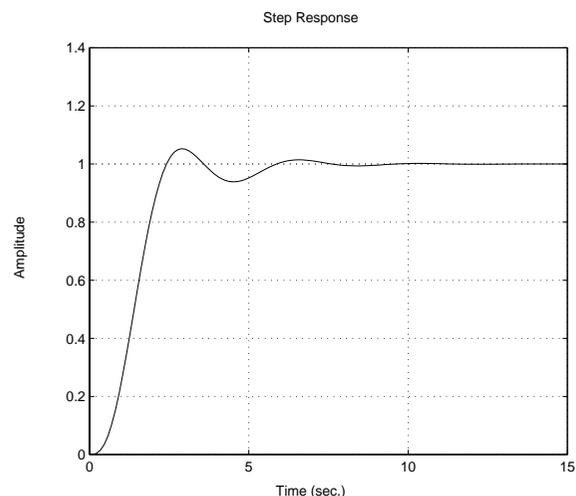
Den vanligaste typen av simulering är att beräkna ett systems stegsvar. Detta kan utföras med funktionen `step`, med vilken man både simulerar systemet och ritar dess stegsvar. I likhet med tidigare kan man läsa av enskilda värden i figuren med vänster musknapp och få en meny med olika val med höger knapp. Genom att t ex välja **Peak Response** från **Characteristics** markeras tidpunkt och värde för överslängen. Placera markören över punkten i diagrammet visas tillhörande numeriska värden.

Antag att systemet  $G$  styrs med proportionell återkoppling med förstärkning  $K_p = 0.7$ .

I normalfallet väljs simuleringstiden automatiskt, men genom att ange ett extra argument kan man välja simuleringstiden själv.

Beräkna det återkopplade systemets stegsvar under femton sekunder och rita upp resultatet.

```
>> step( Gc, 15 )
```



## 7.2 Allmän insignal

För att simulera linjära system med allmänna insignaler kan man använda funktionen `lsim(G,u,t)`. Indata till denna funktion är ett (eller flera) system  $G$ , en insignalvektor  $u$  och en tidsvektor  $t$ .

Antag exempelvis att vi vill studera reglerfelet för det återkopplade systemet ovan då referenssignalen är en ramp. Vi vet att sambandet mellan referenssignal och reglerfel ges av känslighetsfunktionen

$$E(s) = S(s)R(s)$$

där

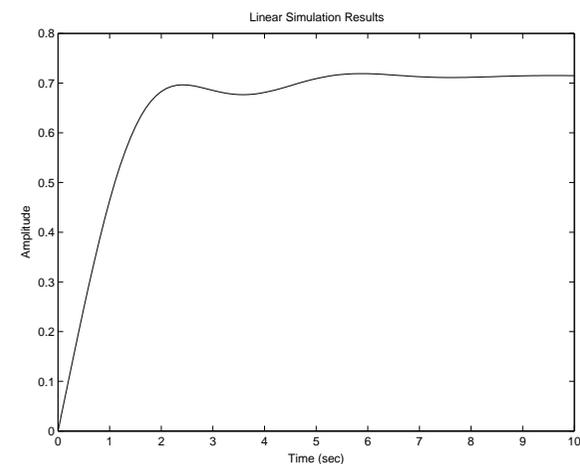
$$S(s) = \frac{1}{1 + F(s)G(s)}$$

Skapa en tidsvektor mellan 0 och 10 med steget 0.1. 

```
>> t = ( 0 : 0.1 : 10 ).';
```

Simulera det återkopplade systemet då referenssignalen är en ramp med lutning 0.5. Reglerfelet går i detta fall mot 0.71. Funktionen ritas även insignalen, men den kan väljas bort på menyn som nås via höger musknapp.

```
>> lsim( S, 0.5*t, t )
```



För att skapa sinus- och fyrkantsignaler kan funktionen `gensig` användas.

## 8 Rotort

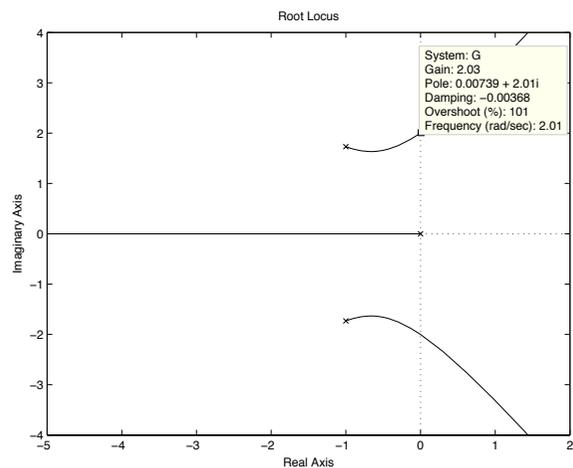
För att avgöra hur rötterna till ekvationen

$$P(s) + KQ(s) = 0$$

rör sig i komplexa talplanet då  $K$  går från noll och mot oändligheten kan man rita ekvationens rotort med funktionen `rlocus`. Indata till funktionen är en överföringsfunktion med polynomet  $Q(s)$  som täljare och polynomet  $P(s)$  som nämnare. Med höger musknapp kan man markera relevanta punkter i figuren, såsom  $t$  ex då rotorten passerar imaginäraxeln.

Rita upp rotorten för det återkopplade systemets karakteristiska funktion då systemet  $G$  styrs med en proportionell återkoppling. Markera där en av rötterna passerar imaginäraxeln.

```
>> rlocus( G )
```



För att t ex kontrollera för vilken förstärkning polerna har viss dämpning kan man med höger musknapp lägga in ett nät vilket markerar polplaceringar med samma avstånd till origo respektive samma dämpning.

## 9 SISO Design Tool

Ett ytterligare användbart verktyg är *SISO Design Tool*, vilket är ett användargränssnitt med vilket man enkelt kan studera ett system ur olika aspekter såsom stegsvar, bodediagram, poler och nollställena, etc. Verket *SISO Design Tool* startas genom att skriva `sisotool`. Automatiskt kommer de skapade LTI-objekten att finnas tillgängliga för analys. I figuren nedan visas ett exempel på vilka figurer som kan visas samtidigt. Testa dig fram!

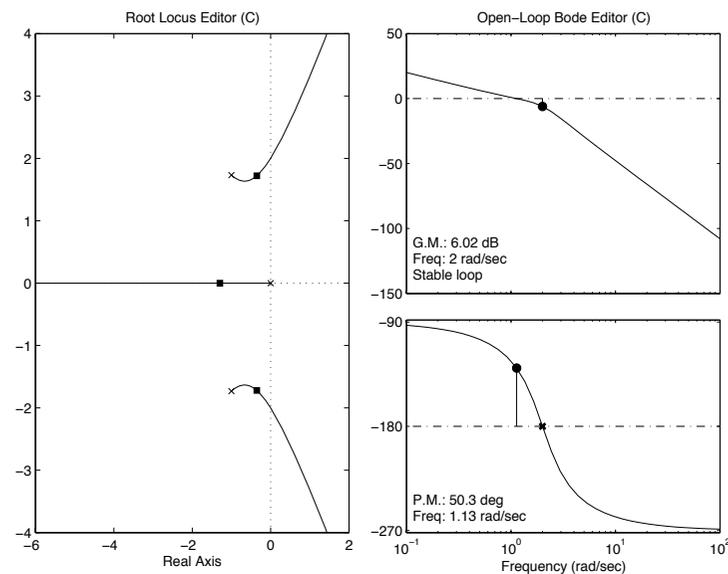


Figure 1. *SISO Design Tool*.

# 10 Tillståndsbeskrivning

I Control System Toolbox finns även en datastruktur för att hantera system på tillståndsform

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

För att skapa ett system på denna form används funktionen `ss`, med vilken man kan skapa ett system på tillståndsform från början eller konvertera ett system från överföringsfunktionsform.

---

Överför systemet `G` till tillståndsform. `>> G = ss( G )`

```
a =
      x1  x2  x3
x1  -2  -2   0
x2   2   0   0
x3   0   1   0
```

```
b =
      u1
x1   2
x2   0
x3   0
```

```
c =
      x1  x2  x3
y1   0   0   1
```

```
d =
      u1
y1   0
```

Continuous-time model.

---

Matriserna  $A, B, C$  och  $D$  i tillståndsbeskrivningen ingår nu i datastrukturen `G`. För att komma åt matriserna kan man referera till dem direkt genom att skriva `G.a`, `G.b` etc.

---

```
Beräkna egenvärdena till ma- >> eig( G.a )
trisen A i tillståndsmodellen
ans =
      0
 -1.0000 + 1.7321i
 -1.0000 - 1.7321i
```

---

Denna möjlighet är användbar t ex när man skall beräkna polplacering tillståndsåterkoppling på formen

$$u(t) = -Lx(t) + r(t)$$

vilket kan göras med funktionen `place` (och i special-fall med `acker`).

---

```
Bestäm en tillståndsåterkop- >> L = place( G.a, G.b, ...
pling som placerar det återkop-      -2 * ( 1 + [ -0.01 0 0.01 ] ) )
plade systemets poler i
närheten av -2. Läger man
alla polerna exakt i -2 kan
återkopplingen inte beräknas
med hjälp av place, så polerna
sprids ut en aning. För att
få alla polerna exakt i -2
kan acker användas, men
den funktionen har så dåliga
numeriska egenskaper att den
alltid bör undvikas till förmån
för place.
```

---

Det återkopplade systemet

$$\dot{x}(t) = (A - BL)x(t) + Br(t)$$

$$y(t) = Cx(t)$$

kan nu skapas t ex med funktionen `ss`.

---

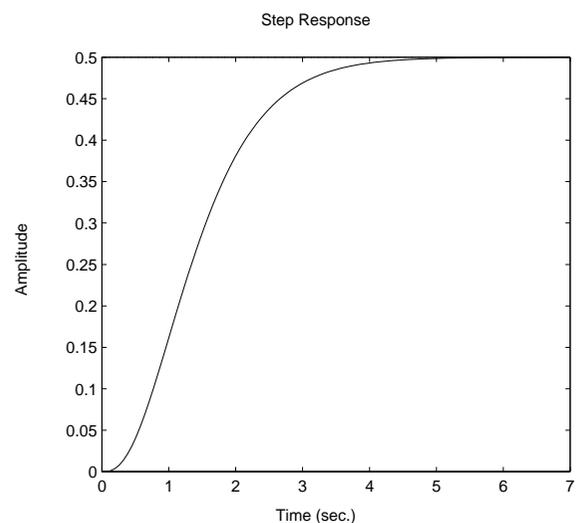
Generera tillståndsbeskrivningen för det återkopplade systemet. Kontrollera att polerna placerats på önskat sätt.

```
>> Gc = ss( G.a - G.b * L, G.b, G.c, 0 );  
>> eig( Gc.a )  
  
ans =  
  
-2.0200  
-2.0000  
-1.9800
```

Det återkopplade systemets stegsvar kan nu beräknas och ritas upp med funktionen `step`.

Beräkna och rita upp det återkopplade systemets stegsvar.

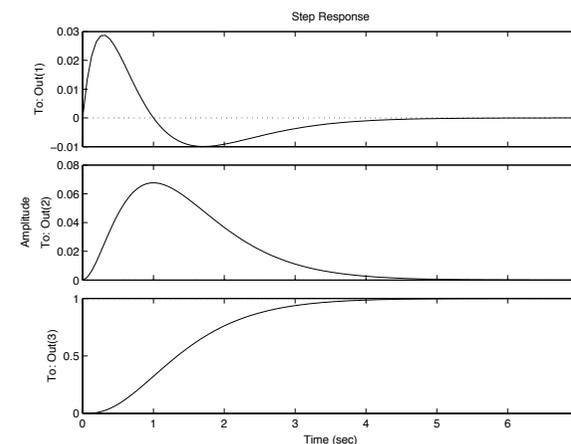
```
>> step( Gc )
```



På detta sätt ser vi endast den utsignal som definieras av vektorn  $C$ . Vill vi studera samtliga tillstånd kan detta göras genom att låta  $C$  vara en enhetsmatris med dimension lika med systemets ordningstal.

Skapa det återkopplade systemet på nytt, men med samtliga tre tillstånd som utsignaler.

```
>> Gc = ss( G.a - G.b * L, G.b, eye(3), 0 );  
>> step( Gc )
```



För att beräkna linjärkvadratisk tillståndsåterkoppling kan funktionen `lqr` användas.

# 11 Sammanfattning av kommandon

## 11.1 Användbara kommandon i Control System Toolbox

<code>tf</code>	System på överföringsfunktionsform
<code>ss</code>	System på tillståndsform
<code>pole</code>	Poler
<code>step</code>	Stegsvar
<code>tzero</code>	Nollställen
<code>feedback</code>	Återkoppling
<code>nyquist</code>	Nyquistdiagram
<code>bode</code>	Bodediagram
<code>bodemag</code>	Bodediagrammets amplitudkurva
<code>sigma</code>	Generalisering av bodemag
<code>margin</code>	Bodediagram och stabilitetsmarginaler
<code>rlocus</code>	Rotort
<code>lsim</code>	Simulering med godtycklig insignal
<code>place</code>	Polplacerande tillståndsåterkoppling
<code>lqr</code>	Linjärvadertisk tillståndsåterkoppling
<code>ctrb</code>	Styrbarhetsmatris
<code>obsv</code>	Observerbarhetsmatris
<code>ltiview</code>	Startar LTI Viewer
<code>pzmap</code>	Pol-nollställediagram
<code>minreal</code>	Förkortning av gemensamma faktorer
<code>sisotool</code>	Grafiskt gränssnitt

## 11.2 Användbara MATLAB-kommandon

<code>abs</code>	Absolutbelopp
<code>eig</code>	Egenvärden
<code>conv</code>	Polynommultiplikation
<code>det</code>	Determinant
<code>diag</code>	Diagonalmatris
<code>imag</code>	Imaginär del
<code>inv</code>	Matrisinvers
<code>real</code>	Realdel
<code>roots</code>	Rötter till polynom
<code>grid</code>	Nät i figurer
<code>hold</code>	Frysning av figur
<code>loglog</code>	Diagram i log-log skala
<code>plot</code>	Diagram i linjär skala
<code>cd</code>	Byte av bibliotek
<code>dir</code>	Listning av bibliotek
<code>clear</code>	Radering av variabler och funktioner i arbetsminnet
<code>load</code>	Inläsning av variabler från fil
<code>save</code>	Lagring av variabler på fil
<code>who</code>	Listning av variabler i arbetsminnet
<code>helpdesk</code>	Startar HTML-baserad hjälpfunktion