Exercise 1.11

a) We would want ~~#~~ C choosen so that if $p \in A$ and $q \in C$ then $pq < 1$ for $p, q > 0$. This should hold if

$$C = \left\{ \frac{1}{r} \, ; \, \text{for } r \in B, \ r \text{ not the least in } B \right\} \cup \{ q \, ; \, q \leq 0 \}$$

$D = \mathbb{Q} \setminus C$.

By definition

$$(A|B) \cdot (C|D) = \left(\{ p \cdot q \, ; \, p \in A, q \in C \text{ and } p, q > 0 \} \cup \{p \leq 0\} \right) \Big| \text{Rest}$$

Now $p \cdot q$ for $p \in A$, $q \in B$ and $p, q > 0$ will satisfy

i) $p \cdot q < 1$ since if $q \in C$ then $\frac{1}{q} \in B$ but if $\frac{1}{q} \in B$ then $p < \frac{1}{q}$ (since all numbers in B are greater than all numbers in A by definition)

ii) $$\underset{\substack{p \in A \\ q \in C \\ p, q > 0}}{\text{l.u.b}} \; p \cdot q = 1$$

This since if $r = \inf_{s \in B} s$ (~~least~~) then we may find a $q \in C$ s.t.

$\frac{1}{q} \in \left(\frac{1}{r} - \varepsilon, r\right)$ and $p \in (r - \varepsilon, r) \subset A$

Thus

$$p \cdot q > \frac{1}{r} \cdot (r - \varepsilon) = 1 - \frac{\varepsilon}{r} \qquad \text{for any} \quad \varepsilon > 0.$$

we can conclude that $\sup_{\substack{p \in A \\ q \in B}} p \cdot q = 1$.

It follows that

$$\{p \cdot q;\ p \in A,\ q \in B,\ p, q > 0\} \cup \{p < 0\} = (-\infty, 1)$$

as desired.

Answer:

$$C|D = \left\{\frac{1}{r};\ r \in B \text{ and } r \neq \text{least element in } B\right\} \cap \{p<0\} \ \Big|\ \text{Rest of } Q$$

Exercise 1.14

We need to show that

$$C = \{ r \in \mathbb{Q} ; \ r = p \cdot q \ \text{for} \ p, q \in A, \ p, q > 0 \ \text{or} \ r \leq 0 \} =$$

$$= \{ r \in \mathbb{Q} ; \ r < 2 \} := D$$

If $r \in C$ and $r > 0$ then $\exists p, q \in A$ s.t. $p \cdot q = r$ and $p, q > 0$.

If we assume that $p \geq q$ ($q > p$ handled similarly) then

$p \cdot q \leq p^2$ but $p \in A$ so $p^2 < 2$

Thus $r = p \cdot q < 2$. It follows that $C \subset D$.

Next we want to show that $D \subset C$

[To show that $C = D$ it is enough to show $C \subset D$ and $D \subset C$]

For that we pick $p \in D$ then, $p < 2$. ~~We know~~ Let $\varepsilon = 2 - p > 0$ and choose n very large, say $n > \frac{8}{\varepsilon}$, then the sequence of rational numbers $q_k = \frac{k}{n}$, $k = 1, 2, \ldots, 2n$

satisfies

$$q_{k+1}^2 - q_k^2 = \frac{2}{n} q_k + \frac{1}{n^2} < \frac{5}{n} < \varepsilon, \quad \text{since } q_k \leq 2$$

and therefore the maximal difference between two consecutive q_k^2 are less than ε.

Let k be the largest natural number s.t.

$q_k^2 \leq p$ then $p < q_{k+1}^2 < p + \varepsilon < 2$

and therefore $q_{k+1} \in A$.

Furthermore $\frac{p}{q_{k+1}} < 1$ so $\frac{p}{q_{k+1}} \in A$.

We may conclude that $p = q_{k+1} \cdot \frac{p}{q_{k+1}}$ is the product of two rational numbers in A. Therefore every $p \in D$ is also contained in C. We conclude that $D \subset C$.

This finishes the proof.

Exercise 1.20.

We will assume that $\lim_{n\to\infty} a_n = b$ and $\lim_{n\to\infty} a_n = b'$, and then use the ε-principle to show that $b = b'$ (for the ε-principle see Thm 8 on p. 21).

We need to show that $|b-b'| < \varepsilon$ for all $\varepsilon > 0$.

Since $a_n \to b$ there exsists an $N_{\varepsilon/2}$ s.t.

$$|a_n - b| < \frac{\varepsilon}{2} \quad \text{for all} \quad n > N_{\varepsilon/2} \qquad (1)$$

and since $a_n \to b'$ there exsists an $N'_{\varepsilon/2}$ s.t.

$$|a_n - b'| < \varepsilon/2 \quad \text{for all} \quad n > N'_{\varepsilon/2}. \qquad (2)$$

Now fix an a_n s.t. $n > N_{\varepsilon/2}$ and $n > N'_{\varepsilon/2}$ then

$$|b-b'| = |a_n - b' + (b - a_n)| \leq \left\{\begin{matrix}\text{triangle}\\ \text{ineq}\end{matrix}\right\} \leq \underbrace{|a_n - b'|}_{< \varepsilon/2 \text{ by } (2)} + \underbrace{|b - a_n|}_{< \varepsilon/2 \text{ by } (1)}$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$