1 The Problem.

Some things you need to know.

- You are not expected to solve this problem!
- If you know how to solve it then this course is not for you. Most likely you have taken a course that covers the material of this course already and the course will bore you. If you haven't taken a course that covers the same material as this course and if you think that you can solve the problem then you are wrong (and welcome to attend the course and learn why) - or you are a genius and should be working on the Riemann hypothesis or something!
- The problem is rather involved and during the course we will develop the theory needed to solve the problem, but it will take most of the term.
- You are supposed to struggle with the problem, realize that it is difficult and learn the hard way why we need to develop abstract analysis.
- Since you are supposed to fail I do not want you to spend to much time on the problem. Take $1\frac{1}{2}$ or 2 hours, try some hypothesis and see where it fails. You should be frustrated but not experience any despair.
- The problem is not well defined, this is on purpose. Part of the problem is to figure out the right formulation - that is the way research works.[1]
- You are not supposed to hand in a solution, but take some notes on how you approached the problem for a class discussion.

Background to the problem: Nature tend to arrange itself so as to minimize an energy. If the function $u(\mathbf{x})$ describes the state of something ($u(\mathbf{x})$ could for instance be the deformation of an elastic body) then the function $u(\mathbf{x})$ will be the function that minimizes a certain well defined energy. Let us consider the simplest possible energy functional:

$$E(u(\mathbf{x})) = \int_{\mathcal{D}} |\nabla u(\mathbf{x})|^2 d\mathbf{x}, \tag{1}$$

where $\mathcal{D}$ is a given subset of $\mathbb{R}^n$ and $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right)$.

In this setting the function $u(\mathbf{x})$ that minimizes (1) is clearly the constant function.[2] So in order to make the problem more interesting (and more physically relevant) we also require that $u(\mathbf{x}) = f(\mathbf{x})$ om the boundary of $\mathcal{D}$, where $f(\mathbf{x})$ is a given function defined on the boundary $\partial\mathcal{D}$.

The Problem: Given a subset $\mathcal{D} \subset \mathbb{R}^n$ and a function $f : \partial\mathcal{D} \mapsto \mathbb{R}$ does it exist a function $u(\mathbf{x})$ such that $u(\mathbf{x}) = f(\mathbf{x})$ on $\partial\mathcal{D}$ and

$$E(u(\mathbf{x})) \leq E(v(\mathbf{x})) \text{for all functions } v(\mathbf{x}) \text{ s.t. } v(\mathbf{x}) = f(\mathbf{x}) \text{ on } \partial\mathcal{D}?$$

[1] One thing with the course is that we will try to motivate the theory we develop from problems we encounter. Unless we know why a concept is needed there is no point learning it. I do not like the approach to mathematics that we have to learn things because they are part of the course book.

[2] Since $E(c) = 0$ for each $c \in \mathbb{R}$ and $E(u) \geq 0$ for all functions u.