What is enumerative combinatorics?

(A). We are given a "description" of a finite set S. How many elements are there in S?

(A'). We are given a sequence $A_1, A_2, A_3, \ldots$ of finite sets. Determine $a_k = |A_k| =$ = number of elements in A_k, $k = 1, 2, 3, \ldots$

What is an answer to (A')?

1). Let A_k be the set of subsets of $\{1, 2, \ldots, n\}$. Then $a_k = 2^k$.

2). Let $A_{n,k}$ be the set of k-element subsets of $\{1, 2, \ldots, n\}$. Then

$$|A_{n,k}| = \binom{n}{k} = \frac{n!}{k!\,(n-k)!},$$

where $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$

3). How many partitions of n are there? A partition of n is a sequence of nonnegative integers $(\alpha_1, \ldots, \alpha_n)$ s.t. $\alpha_1 + \alpha_2 + \ldots + \alpha_n = n$ and $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0$. Call this number $p(n)$.

<u>Euler</u>: $$\sum_{n=0}^{\infty} p(n) x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

<u>Hardy-Ramanujan</u>: $$p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}$$

An answer to (A) or (A') could be:

(1). Exact formulas

(2). Asymptotic formulas

(3). Generating functions

(4). Express one in terms of another (implicit answer).

- We will also be interested in the structure of combinatorial models. Primarily <u>posets</u>, i.e., partially ordered sets.
- <u>Bijections</u>: How do we prove $|A| = |B|$?

The "best" way is to find a "simple" bijection $f: A \to B$.

<u>Basic principles</u>:

- Recall $A \times B := \{(a,b) : a \in A, b \in B\}$

  $|A \times B| = |A| \cdot |B|$. (Multiplication principle).
- If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$

  otherwise $|A \cup B| = |A| + |B| - |A \cap B|$.

<u>Example</u>: Let S be a finite set, $|S| = n$, $S = \{s_1, s_2, \ldots, s_n\}$. Define $2^S := \{T : T \subseteq S\}$.

Then $|2^S| = 2^n$. Find bijection $f: 2^S \to \underbrace{\{0,1\}}_{1} \times \underbrace{\{0,1\}}_{2} \times \cdots \times \underbrace{\{0,1\}}_{n}$

$f(T) = (X_1(T), X_2(T), \ldots, X_n(T))$ where

$$X_i(T) = \begin{cases} 1 & \text{if } s_i \in T \\ 0 & \text{otherwise.} \end{cases}$$

- A <u>permutation</u> of a finite set S is a bijection $\pi : S \to S$. Write $\mathfrak{S}(S) := \{\pi : \pi \text{ is a permutation of } S\}$

If $S = \{1, \dots, n\}$ we write $\mathfrak{S}(S) = \mathfrak{S}_n$.

If $\pi \in \mathfrak{S}_n$ we write $\pi = \pi_1 \pi_2 \dots \pi_n$, where $\pi_i = \pi(i)$, for example $\pi = 41532$

- Clearly $|\mathfrak{S}_n| = n!$:

n choices for π_1
$n-1$ —//— π_2
$n-2$ —//— π_3
...

- Prove this using a bijection:

If $a \leq b$ are integers, let $[a, b] := [a, a+1, \dots, b-1, b]$

we want to find a bijection

$$I : \mathfrak{S}_n \longrightarrow [0, n-1] \times [0, n-2] \times \dots \times [0, 1] \times [0, 0]$$

$(=: B_n)$

Define $I(\pi) = (b_1, b_2, \dots, b_n)$, where

b_i = # j's in π s.t. $j > i$ and j is to the left of i in π.

$$I(41532) = (1, 3, 2, 0, 0)$$

Clearly $b_i \leq n - i$ so that $I(\pi) \in B_n$.

Find inverse: Build π from $(1,3,2,0,0)$ by placing out $\{1, 2, \dots 5\}$ in the order $5, 4, 3, 2, 1$

(1). Place 5 : 5

(2). Place 4 : 45 (since 4 has no larger entry to its left)

(3). Place 3 : 453 (since 3 has 2 larger entries to its left).

(4). Place 2 : 4532

(5). Place 1 : 41532

This algorithm determines the inverse

Binomial numbers: Let $[n] := \{1, 2, \dots, n\}$.

$$\binom{n}{k} := \left|\{ S : S \subseteq [n], |S| = k \}\right| = \left|\binom{[n]}{k}\right|$$

Thm: $$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

(note that for fixed k, the function $n \to \binom{n}{k}$ is a polynomial of degree k)

Proof: We want to prove $k!(n-k)!\binom{n}{k} = n!$.

Find a bijection

$$\varphi : \mathfrak{S}_k \times \mathfrak{S}_{n-k} \times \binom{[n]}{k} \longrightarrow \mathfrak{S}_n$$

$\varphi(\sigma, \tau, T) = \pi$, where the first k entries of π are T, the $n-k$ last entries are then $[n] \setminus T$, the first k entries are ordered as σ, the last $n-k$ — " — τ.

$$\varphi(231, 3142, \{3,4,6\}) = 463\,5172$$

Clearly a bijection. $\square$

Pascal's triangle: $$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\binom{[n]}{k} = \binom{[n-1]}{k} \,\dot\cup\, \left\{ S \cup \{n\} : S \in \binom{[n-1]}{k-1} \right\}$$

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          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5  10  10   5   1
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Binomial theorem: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^n$

Let $x_1, x_2, \ldots, x_n$ be independent variables.
When expanding $(1+x_1)(1+x_2)\cdots(1+x_n)$
we either have to choose 1 or x_i from the
i'th paranthesis:

$$(1+x_1)\cdots(1+x_n) = \sum_{S \subseteq [n]} \prod_{j \in S} x_j$$

Set $x_j = x$ for all j.

. A k-composition of n is a vector
$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ of positive integers s.t.

$$\sigma_1 + \cdots + \sigma_k = n.$$

Let $\mathcal{C}_{n,k} = \{k\text{-comp. of } n\}$

Ex: $(6,2,2,3,1,5)$ is a 6-composition of 19.

Fact: $|\mathcal{C}_{n,k}| = \binom{n-1}{k-1}$

Proof: Define $\varphi : \mathcal{C}_{n,k} \to \binom{[n-1]}{k-1}$

$$\varphi(\sigma_1, \ldots, \sigma_k) = \{\sigma_1, \sigma_1+\sigma_2, \ldots, \sigma_1+\sigma_2+\cdots+\sigma_{k-1}\} \in \binom{[n-1]}{k-1}$$

Now $\varphi^{-1}(S) = (s_1, s_2 - s_1, s_3 - s_2, \ldots, s_{k-1} - s_{k-2}, n - s_{k-1})$

where $S = \{s_1, s_2, \ldots, s_k\}$ and $s_1 < s_2 < \cdots < s_k$. $\square$