Exercise 22 chapter 1

a) If the graph $\{(x,f(x));\ x\in A\} \cap \{(x,x);\ x\in A\} \neq \phi$

then there exist some $x_0\in A$ s.t.

$(x_0, f(x_0)) \in \{(x,x);\ x\in A\}$, that is $(x_0,f(x_0))=(x_0,x_0)$

and thus $f(x_0)=x_0$.

If $f(x_0)=x_0$ for some $x_0\in A$ then

$(x_0,x_0)\in\{(x,f(x)),\ x\in A\}$ and thus the intersection with the diagonal is non-empty.

b) If $f(0)=0$ or $f(1)=1$ then we are done.

So let us assume that $f(0)>0$ and $f(1)<1$.

Then $f(x)-x=g(x)$ satisfies $g(0)>0$ & $g(1)<0$.

The statement follows by the intermediate value Thm.

But let us indicate the argument.

The set $S=\{x\in[0,1];\ g(x)>0\}$ is non-empty (since $0\in S$) and bounded from above (by $x=1$).

Therefore $l.u.b(S)=x_0$ exsists.

Since $x_0-\frac{1}{k}$ is not a l.u.b of S $\exists x_k\in[0,1]$ s.t. $x_k\in(x_0-\frac{1}{k}, x_0)$ and $g(x_k)>0$. Also

since x_0 is a least upper bound of S $g(x_0+\frac{1}{k})<0$.

By continuity and the thm about inequalities in passing to the limit we can conclude that

$$0 \leq \lim_{k \to \infty} g(x_k) = g(x_0) = \lim_{k \to \infty} g(x_0 + \tfrac{1}{k}) \leq 0,$$

since $x_k \to x_0$ and $x_0 + \frac{1}{k} \to x_0$ and continuity of $g(x)$. Thus $0 = g(x_0) = f(x_0) - x_0 \Rightarrow f(x_0) = x_0$.

c) No it is not. consider for instance

$f(x) = \sin\left(\frac{\pi x}{2}\right)$ or simpler

$f(x) = \frac{1}{2} + \frac{x}{2}$ which is strictly larger than

x on $(0,1)$.

d) No, consider

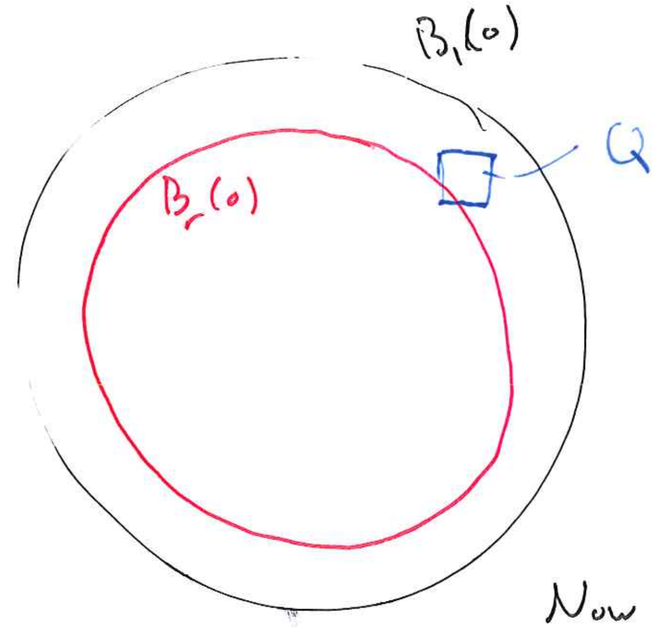
$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < x \leq 1. \end{cases}$$

Exercise 25a chapter 1.

Let $B_r(0) = \{x \in \mathbb{R}^2 ;\ |x| < r\}$ be the ball of radius r in $\mathbb{R}^2$. Then the area of $B_1(0) \setminus B_r(0)$ is $\pi(1-r^2)$ and we may choose $r < 1$ s.t. $\pi(1-r^2) < \varepsilon$.

Next we consider the dyadic cubes with side $\frac{1}{2^n}$ and corners in $\left(\frac{p}{2^n}, \frac{q}{2^n}\right)$ for $p, q \in \mathbb{Z}$.

We may choos n large enough so that $\frac{1}{2^n} < \frac{(1-r)}{2}$. Then any cube Q that intersect $B_r(0)$ will be entirely contained in $B_1(0)$



since the distance from $B_r(0)$ to the compliment of $B_1(0)$ is $(1-r)$ which is larger than the length of the diagonal of Q (which is $\frac{\sqrt{2}}{2^n} < (1-r)$).

Now consider the cubes Q_k that intersect $B_r(0)$: that is $\bigcup_{Q_k \cap B_r(0) \neq \emptyset} Q_k$ then

$B_r(0) \subset \bigcup_k Q \subset B_1(0)$ and thus the area of $\bigcup_k Q_k$ must be larger than πr^2. Since $\pi(1-r^2) < \varepsilon$ the statement follows.

Ex 31 chapter 1

a) If $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$ (1) then f has to be continuous at x_0 since, for $h > 0$,

$$f(x_0 - h) \leq f(x_0) \leq f(x_0 + h) \qquad [f \text{ is non-decreasing}].$$

thus $\lim_{h \to 0^+} f(x_0 - h) \leq f(x_0) \leq \lim_{h \to 0^+} f(x_0 + h)$

and by (1) the left and right sides are equal, and we know that if the left and right limits agree with the functions value at a point then it is continuous at that point.

Next we notice that, with $f(b) - f(a) = M$, The set

$$D_k = \{x ;\ f(x_k^+) - f(x_k^-) > \tfrac{1}{k}\}$$ must have

cardinality less then kM.

Indeed

$$M = f(b) - f(a) = \underbrace{\left(f(b) - f(x_N^+)\right)}_{\geq 0} + \underbrace{\left(f(x_N^+) - f(x_N^-)\right)}_{> 1/k} + \underbrace{\left(f(x_N^-) - f(x_{N-1}^+)\right)}_{\geq 0}$$

$$+ \underbrace{\left(f(x_{N-1}^+) - f(x_{N-1}^-)\right)}_{\geq \frac{1}{k}} + \dots + \underbrace{\left(f(x_1^-) - f(a)\right)}_{\geq 0} = N/k$$

if $x_1 < x_2 < \dots < x_N$ and $x_1, x_2, \dots, x_N \in D_k$

where we used that $f(x)$ is non-decreasing in estimating $f(x_j^-) - f(x_{j-1}^+) \geq 0$.

Since each D_k is finite the set of discontinuities $\bigcup_{k=1}^{\infty} D_k$ must be denumerable ~~countable~~ by Corollary 18.

b) Since $\mathbb{R} = \bigcup_k [-k, k]$ and, by part a, the set of discontinuities of $f(x)$ in $[-k, k]$ is denumerable it follows that the set of discontinuities on $\mathbb{R}$ = (union of the discontinuities on $[-k, k]$) is the ~~countable union of~~ denumerable union of denumerable sets which again is denumerable by Corollary 18.