Course plan SF1677

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1 Lecture 1.

In this first lecture we will *construct the real numbers*. The properties of the real numbers are at the heart of mathematical analysis. In particular the *least upper bound property*. But it is very unsatisfactory to just assume that the real numbers satisfy a property without a proof - or without even knowing what the real numbers are. How do we know that we will not be able to derive a contradiction if we assume the least upper bound property? The easiest way to know that is to construct the real numbers from something we know well: the rational numbers.

The most important thing that you should know after this lecture is the construction of the real numbers and the importance of the least upper bound property. You should be able to prove Theorem 2 and Theorem 5.

Self study: (will not be discussed during the lecture) Chapter 1.1

Reading: Chapter 1.2

Suggested exercises: 1 (see ex 0 first), 2, 9, 10, **11**, 12, 13, **14**, **17a**, **20** The most important exercises are in **bold**.

2 Lecture 2.

In this lecture we will introduce two things: Eucledian spaces and some theory of the infinite. There are several important things that you need to know. First of all you need to know the concepts introduced (in bold face in Pugh). There are several important concepts in the reading for this lecture - and you should know all concepts. The most important proofs are of the Cauchy-Schwartz and triangle inequalities in section 1.3. In section 1.4 the most important Theorems are that $\mathbb{R}$ (Thm 10) is uncountable and $\mathbb{Q}$ is countable (Cor 19). Learn the proofs of these results!

Reading: Chapter 1.3, 1.4

Suggested exercises: 22, 24, 25a, 28, 31, 35

3 Lecture 3.

This lecture covers one difficult theorem (The Bernstein-Schroeder Theorem) as well as a repetition of first year calculus. You should know the entire section 1.6

(Continuous functions stuff) by heart. Try to read and understand the proof of the Bernstein-Schroeder Theorem, but you are not required to memorize it.

We will also define a metric space - this is a central thing for all analysis and you should know the definition. Also Learn the proof of Theorem 1 in section 2.1.

Reading: Chapter 1.5, 1.6 and 2.1

Suggested exercises: Chapter 1: 36a (difficult), 37 (difficult), **41, 42, 43, 44** Chapter 2: 3, 4, 5

4 Lecture 4.

This lecture contains very much important material and you should know most of the reading by heart. Fortunately, most proofs are rather simple once you understand them.

Parts of the reading is repetition of theory covered in the first undergraduate course in analysis and you should make sure that you know those results. The definition of continuity used by Pugh is not the normal definition (the normal definition is stated as Theorem 4 which you should know.).

The concept of a **Homeomorphism** is very fundamental for all topology so make sure that you know its definition. Other important concepts that you should know are **open/closed** sets. The most important theorems relating to Topology are Theorem 6 and 7 with their corollaries (Corollary 7 and 12). Make sure that you know them.

Reading: Chapter 2.2 and 2.3 (up to page 73)

Suggested exercises: Chapter 2: **9, 10,** 12, **14**, 20, 23, **26**, 28, 31

5 Lecture 5.

Lecture 5 also covers much very important material and you must know most of it - I wish that I could pick a few results and claim that those where the most important results; but I have to stress most of the results as important. The good thing is that the theory is based on rather few and intuitive principles and most people can master the theory after spending some time with it.

Two very important concepts are **Completeness** and **Compactness**. Therefore you need to know Theorems 23 ($\mathbb{R}^n$ complete), 26, 27 and 28 (they are similar but you will need them). Theorem 31 (Bolzano-Weierstrass) and 33 (Heine-Borel) are also very important and you should know them. Notice that Theorems 26 and 33 implies the other important theorems stated in this paragraph.

Another classical theorem from basic analysis is Theorem 42 which you should know.

There are also some important topological results this week: in particular Theorem 36 (with its Corollary 37, which again is an old friend from first year calculus), and Theorem 38 with its Corollary 39.

Corollary 39 is rather interesting because it contains one of the basic ideas of topology. The main topic of topology is tu understand which spaces are homeomorphic. And the main idea in to find properties that are preserved under homeomorphisms. If two spaces, for instance $\mathbb{R}$ and $[0, 1]$, does not share a property that is preserved under homeomorphisms then they cannot be homeomorphic. In advanced topology many properties[1] are invented in order to distinguish between spaces that are not homeomorphic, and at times one can say that two spaces are homeomorphic if they share a certain number of invariants. In any case, Corollary 39 is the first example of a technique that almost defines topological research today.

Reading: Chapter 2.3 (the rest) and 2.4

Suggested exercises: Chapter 2: **35**, 36, 37, **38**, 39, 42, **43**, 44, **46**, 52, **56**

6 Lecture 6.

The main topic of this lecture is **connectedness**, which you should know. Also learn Theorem 43 with its important Corollary 45. Theorems 46 and 51 are also among the most important of the chapter.

Reading: Chapter 2.5

Suggested exercises: Chapter 2: **66, 67**, 71, 72, 73, **76**, 77, **78**, 84

7 Lecture 7.

Section 2.6 contains some basic things, most of it from undergraduate calculus, that you should know already but repeat it if you don't. The most important Theorem this week is Theorem 63 from section 2.7, covering compactness is often used as the definition of compactness. The Lebsegue number Lemma 64 is surprisingly useful.

Reading: Chapter 2.6 and 2.7

Suggested exercises: Chapter 2: **87, 89**, 91, **95ab**, 96

At the latest you will be given a set of homework exercises to be handed in on lecture 9 this week!

8 Lecture 8.

Section 2.8 and 2.9 contains much fun stuff, but it is not central to analysis. You should know about the Cantor set since it it a standard source for nice counterexamples in analysis; *the Devil's staircase* that we will meet later in the course is one of my favorite nice examples and its construction is based on the cantor set.[2]

Probably the most important Theorem is sections 2.8 and 2.9 is Theorem 72 about space filling curves. This Theorem is more of an example of a nasty function that is defined on an interval but its image is the entire unit disc in $\mathbb{R}^2$. So why is this interesting? Well a naive definition of the dimension of a space $\mathcal{S}$ would be that a space has dimension n if we need n real numbers to specify any point in $\mathcal{S}$. This is how dimension is approached in linear algebra

[1] An property preserved under homeomorphisms are usually called a topological invariant.

[2] The Devil's staircase is a non-constant continuous function on $[0, 1]$ that has zero derivative at almost all its points.

for instance. But the example in Theorem 72 shows that this definition would imply that the unit disc in $\mathbb{R}^2$ would be one dimensional - $\mathbb{R}$ and the disc in $\mathbb{R}^2$ and even $\mathbb{R}$ and $\mathbb{R}^2$ would have the same dimension! Somehow the concept of definition is deeper than one might think initially. This leave the question of how one defines two spaces to have the same dimension. This leads to the concept of homeomorphism that we have encountered. It turns out that $\mathbb{R}^n$ and $\mathbb{R}^m$ are only homeomorphic if $n = m$ and we can therefore define the dimension of a space by using homeomorphisms. As with all concepts in mathematics, homeomorphism is defined the way it is defined because it helps us understand fundamental and important properties.

Section 2.10 contains a general construction of completeness that is abstract and very powerful. We did construct the complete space $\mathbb{R}$ from the incomplete space $\mathbb{Q}$ by means of Diedekind cuts. But there is a much more general construction where one can form complete spaces from incomplete spaces in many different circumstances. The spaces in questions could be spaces of functions, spaces of other spaces, spaces of differential operators or pretty much anything where we can define a metric. Make sure that you understand Theorem 80.

Reading: Chapter 2.8, 2.9 and 2.10

Suggested exercises: Chapter 2: 103, *This week you should be working on the homework.*

9 Lecture 9.

This lecture is a repetition of the analysis in one variable course. Fortunately for you you already know all the material in this section - or else you will have to learn/repeat it.

Reading: Chapter 3.1

Suggested exercises: Chapter 3: **1, 2, 3, 4**, 5, 8, 9, **12**, 14, **15**

10 Lecture 10.

This lecture is also mostly repetition of calculus. One theorem is new and difficult: The Riemann-Lebesgue Theorem. The Theorem is important and you should know its proof.

Reading: Chapter 3.2 (to p.179)

Suggested exercises: Chapter 3: **19, 26, 27, 28,** 29

11 Lecture 11.

The consequences of the Riemann-Lebesgue Theorem are all important, but simple. Then we repeat some theory from first year calculus again such as the fundamental Theorem of calculus - very important stuff that you know already.

Reading: Chapter 3.2 (to p. 186)

Suggested exercises: Chapter 3: **32**, 33 (difficult), 34 (difficult), **39**, **47**, 48

12 Lecture 12.

You should also Theorem 37. This theorem shows that we do not really understand everything about derivatives. We think that derivatives describes how a function changes; and the fundamental Theorem of Calculus shows that this is indeed the case. In particular, if $f'(x)$ is continuous on (ab) then it determines the difference of $f(x)$ at any two points $x_0, x_1 \in (a, b)$ by $f(x_1) - f(x_0) = \int_{x_0}^{x_1} f'(x)dx$. But the Devil's staircase is a function that is differentiable except at a zero set and the derivative is zero everywhere where it is defined - but yet the function is not constant. This, and similar, examples led to many deep investigations in the second half of the 19th century before one could gain a better understanding of the fundamental Theorem of calculus and when it is applicable. In this course we will not study any of these developments. But Theorem 37 is still important since it shows that there is more to the Fundamental theorem of Calculus than one might think at first.

Section 3.3 contains some theory for infinite series. Some stuff you know already but some of it is probably new. You should know and be able to prove convergence of series by the comparison, integral, root and ratio tests.

Reading: Chapter 3.2 (the rest) and 3.3

Suggested exercises: Chapter 3: **39**, **40**, **41**, 43, **51**, 52, 53, 54, 58, 62,

13 Lecture 13.

Reading: Chapter 4.1

Suggested exercises: More prelim problems Chapter 4: **1**, 3, **4**, 5, **6**, 7, 8, **9**, 11, **13**, 18

14 Lecture 14.

Reading: Chapter 4.2 and 4.3

Suggested exercises: More Prelim problems Chapter 4: **2, 15**, 20, 21, 24, 30, 45

15 Lecture 15.

Reading: Chapter 4.4

Suggested exercises: More Prelim problems Chapter 4: 31, 33, 35, 37, 39

16 Lecture 16.

Reading: Chapter 4.5 and 4.6

Suggested exercises: Chapter 4: **34, 35, 36**

Suggested exercises: More Prelim problems Chapter 4: **38**

17 Lecture 17.

Reading: Chapter 4.7

Suggested exercises: More Prelim problems Chapter 4: **32**
Add Homework!

18 Lecture 18.

Reading: Chapter 5.1 and 5.2 (to p. 285)

Suggested exercises: Chapter 5: **1, 2**, 3, 4, 8, 15

19 Lecture 19.

Reading: Chapter 5.2 (the rest) and 5.3

Suggested exercises: Chapter 5: **16, 17, 18, 20**, 22, 23a, 24

20 Lecture 20.

Reading: Chapter 5.4

Suggested exercises: Chapter 5: **30**, 33, 36

21 Lecture 21.

Reading: Chapter 5.5

Suggested exercises: Chapter 5: **43**

22 Lecture 22.

Reading: Chapter 5.6 and 5.7 (to p. 319)

Suggested exercises: Chapter 5: 44, **45**, **46**, 47

23 Lecture 23.

Reading: Chapter 5.7

Suggested exercises: Chapter 5: No exercises this week!

24 Lecture 24.

Reading: Chapter 5.8 (to p. 334)

Suggested exercises: Chapter 5: **50**, **55**, 57

25 Lecture 25.

At the lecture, Chapter 5.8 (the rest)

Suggested exercises: Chapter 5: **61**, **67**

26 Lecture 26.

Reading: Chapter 5.9

Suggested exercises: Chapter 5: **58**, 68

27 Lecture 27.

Reading: Chapter 5.10

Suggested exercises: Chapter 5: 62

28 Final Exam 10th January 8-13.