1.1

(a) Y and X are independent $\Rightarrow f_{XY}(x,y) = f_X(x) f_Y(y)$

$\Rightarrow E[XY] = E[X]E[Y]$

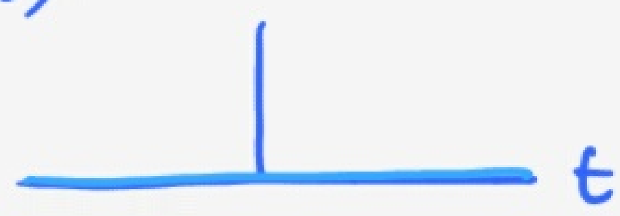
Additionally $P(X=0) = P(X=1) = P(Y=0) = P(Y=1) = 1/2$

$Z(t) = (10-t)X + tY$

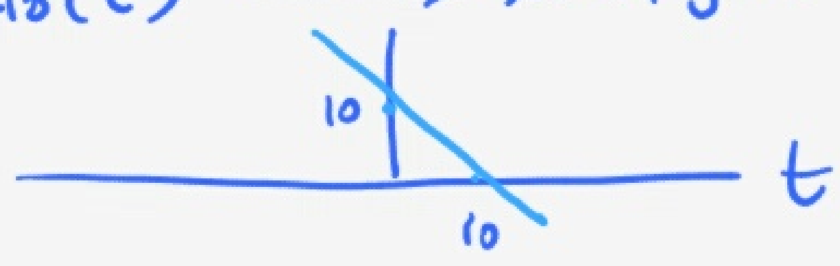
Then all possible realizations (deterministic functions of time):

$Z(t)$	for $X=0$	for $X=1$
for $y=0$	0	$10-t$
for $y=1$	t	10

$z_{00}(t)$



$z_{10}(t) \longrightarrow x=1, y=0$



$z_{01}(t) \to x=0, y=1$

$z_{11}(t)$

(b) $r_Z(t_1,t_2) = E[Z(t_1)Z(t_2)] = E[((10-t_1)X + t_1Y)((10-t_2)X + t_2Y)] =$

$= E[(10-t_1)(10-t_2)X^2 + (10-t_1)t_2XY + t_1(10-t_2)YX + t_1t_2Y^2] =$

$= \left\{ \begin{array}{l} E[\cdot] \text{ is a linear operator} \\ \text{Quantities } t_1 \text{ and } t_2 \text{ are deterministic} \end{array} \right\} = (10-t_1)(10-t_2)E[X^2] + (10-t_1)t_2E[XY] +$

$+ t_1(10-t_2)E[YX] + t_1t_2E[Y^2] =$

$E[X^2] \to \left\{ \begin{array}{l} E[g(x)] = \sum_i g(x)p(x) \text{ (discrete } x) \\ \text{so } E[X^2] = \sum_i x^2 p(x) = \\ = (1)^2 \frac{1}{2} + (0)^2 \cdot \frac{1}{2} = 1/2 \end{array} \right\}$

$E[X^2] = E[Y^2] \Rightarrow$ same values and same probabilities.

$E[XY] \Rightarrow \left\{ X \text{ and } Y \text{ are independent} \right\} \to E[XY] = E[X]E[Y]$

$E[X] = \sum_i x p(x) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

$= \frac{1}{2}(10-t_1)(10-t_2) + \frac{(10-t_1)t_2}{4} + \frac{t_1(10-t_2)}{4} + \frac{t_1t_2}{2} =$

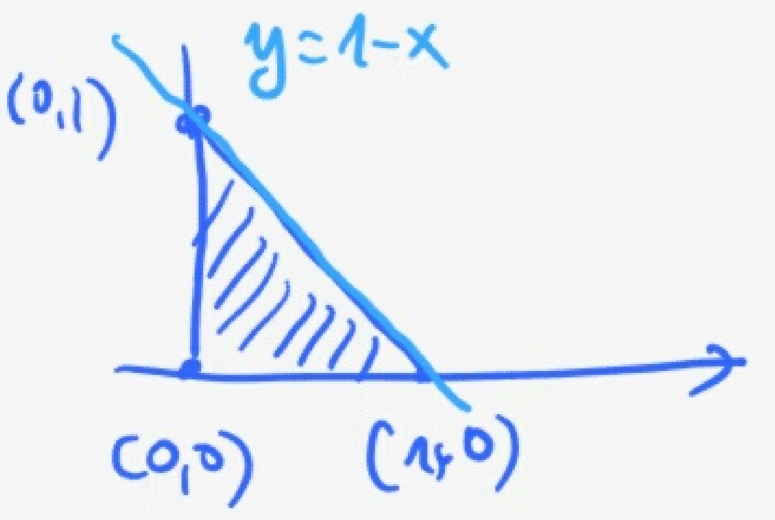
$= \frac{1}{2}\left(10 - t_1 + \frac{t_1}{2}\right)(10-t_2) + \frac{(10-t_1+2t_1)t_2}{4} =$

$= \frac{1}{2}\left(10 - \frac{t_1}{2}\right)(10-t_2) + \frac{(10+t_1)t_2}{4} = \frac{100}{2} - \frac{10}{2}t_2 - \frac{5t_1}{2} + \frac{t_1t_2}{4} +$

$+ \frac{10t_2}{4} + \frac{t_1t_2}{4} = 50 - \frac{5t_1}{2} + \left(\frac{10}{4} - \frac{10}{2}\right)t_2 + \frac{2t_1t_2}{4} = 50 - \frac{5t_1}{2} - \frac{5}{2}t_2 + \frac{t_1t_2}{2}$

$\boxed{1.14}$

$f_{xy}(x,y)$ is uniform on



$\Downarrow$

$f_{xy}(x,y) = c$ within the triangle. A probability density function needs to fulfill that $\iint f_{xy}(x,y)\,dx\,dy = 1$.

and for this case is $\int_0^1 dx \int_0^{1-x} c\,dy = 1$

$$c\int_0^1 (1-x)\,dx = c\left(x - \frac{x^2}{2}\right)_0^1 = c\left(1-\frac{1}{2}\right) = \frac{1}{2}c = 1 \Rightarrow \boxed{c=2}$$

Hence, $f_{xy}(x,y) = 2$ within the triangle and $\emptyset$ outside.

- $r(X,Y) = E[XY] = \iint xy\, f_{xy}(x,y)\,dx\,dy = \int_0^1\int_0^{1-x} 2xy\,dx\,dy = 2\int_0^1 x\,dx\left[\frac{y^2}{2}\right]_0^{1-x} =$

$$= \cancel{2}\int_0^1 \frac{x(1-x)^2}{\cancel{2}} = \int_0^1 (x - 2x^2 + x^3)\,dx = \left[\frac{x^2}{2} - 2\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 =$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{6-4\cdot 2+3}{12} = \frac{1}{12} \quad //$$

- $C(X,Y) = r(X,Y) - m_x m_y = \frac{1}{12} - m_x m_y$.

$m_x = E[X] = \int x\, f_x(x)\,dx$

$$f_x(x) = \int f_{xy}(x,y)\,dy = \int_0^{1-x} f_{xy}(x,y]\,dy = \int_0^{1-x} 2\,dy = 2(1-x)$$

depending on which value of x we select, y has a diferent Range of possible values

Additionally we know that $y = 1-x \Rightarrow x = 1-y$, hence:

$$f_y(y) = \int_0^{1-y} f_{xy}(x,y)\,dx = 2(1-y)$$

Since they have the same distribution for the exact same values. i.e. $x \in [0,1]$ and $y \in [0,1] \Rightarrow m_x = m_y$.

$$E[X] = \int_0^1 x\,2(1-x)\,dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = 2\cdot\frac{1}{6} = \frac{1}{3}$$

$$C(X,Y) = \frac{1}{12} - \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{12} - \frac{1}{9} = \frac{3-4}{36} = \frac{-1}{36}$$

• $\rho(X,Y) = \dfrac{C(X,Y)}{\sigma_x \sigma_y}$ $\qquad C(X,Y) = -1/36$.

$\sigma_x = \sqrt{E[(X-m_x)^2]}$

$E[(X-m_x)^2] = E[X^2] - m_x^2 = \int x^2 f_x(x) - \frac{1}{9} = \int_0^1 x^2\, 2(x-1)dx - \frac{1}{9} =$

$= \left[\frac{2x^4}{4} - \frac{x^3}{3}\right]_0^1 - \frac{1}{9} = \frac{2}{4} - \frac{1}{3} - \frac{1}{9} = \frac{18-12-4}{36} = \frac{2}{36} = \frac{1}{18}$

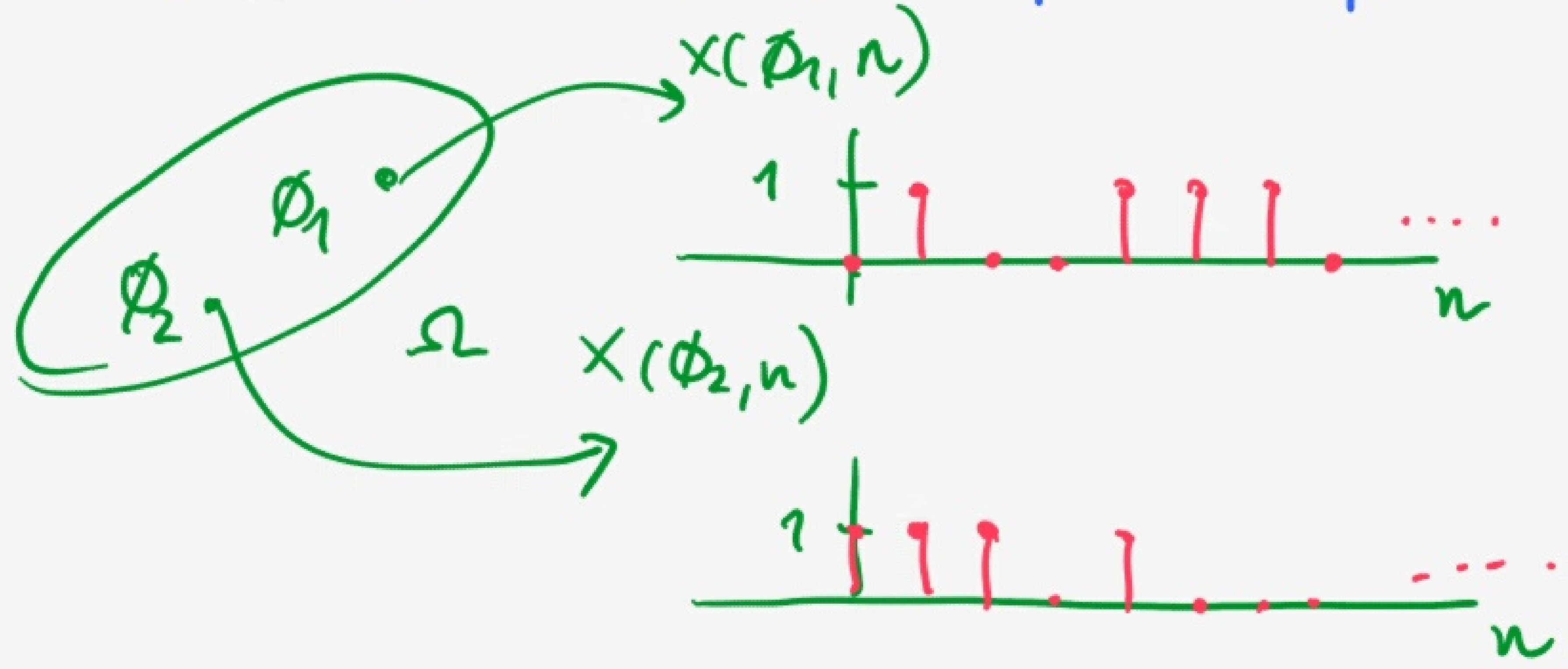
$\sigma_x^2 = \sigma_y^2$ since x and y follow identical p.d.f.'s.

Hence $\sigma_x \sigma_y = \frac{1}{18}$

$\rho(X,Y) = \dfrac{-1/36}{1/18} = -\dfrac{18}{36} = -1/2$

$\boxed{2.2}$ $X(n)$ is a stationary stochastic process

• each realisation is a sequence of independent 1's and 0's, where 1's and 0's have the same probability



Sequence of independent 1's and 0's means that $X(n_1)$ and $X(n_2)$ are independent random variables

i.e. $E[X(n_1)X(n_2)] =$
$= E[X(n_1)]E[X(n_2)]$

We also know that the process is stationary, hence $E[X(n_1)] = m_x$

$E[X(n_2)] = m_x$

$\vdots$

$E[X(n_m)] = m_x \quad \forall n$

About the random variables we know that they take 0 and 1 with probability 1/2.

Hence $E[X(n)] = \frac{1}{2}\ \forall n$ and $E[X(n_1)X(n_2)] = E[X(n_1)]E[X(n_2)]$

which is equivalent to $E[X(n+k)X(n)] = E[X(n+k)]E[X(n)]$

$(k \neq 0)$

(a) $m_Y = E[Y(n)] = E\left[\frac{1}{2}(X(n) + X(n-1))\right] = \Longrightarrow$ {Linear and constants get out}

$= \frac{1}{2}\left(E[X(n)] + \underbrace{E[X(n-1)]}_{\substack{=E[X(n)] \\ \text{(stationary!)}}}\right) = \frac{2}{2} E[X(n)] = \frac{1}{2}$

(b) $P_Y = E[Y(n)^2] = \frac{1}{4} E\left[(X(n) + X(n-1))^2\right] =$

$= \frac{1}{4}\left(E[X^2(n)] + 2X(n)X(n-1) + \underbrace{E[X(n-1)^2]}_{\substack{=E[X(n)^2] \\ \text{(stationarity!)}}}\right) =$

$= \frac{1}{4}\left(2E[X^2(n)] + 2E(X(n)X(n-1)]\right) = \frac{1}{2}E[X^2(n)] + \frac{1}{2}E[X(n)]\underbrace{E[X(n-1)]}_{=E[X(n)]} =$

samples are independent

$= \frac{1}{2}E[X^2(n)] + \frac{1}{2}E[X(n)]^2 = \left\{ \begin{array}{l} E[X^2(n)] = \frac{1}{2}(1)^2 + \frac{1}{2}(0) \\ E[X(n)] = 1/2 \end{array} \right\} =$

$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{4} + \frac{1}{8} = \frac{2+1}{8} = \frac{3}{8}$

- variance σ_Y^2

$\sigma_Y^2 = P_Y - m_Y^2 = \frac{3}{8} - \left(\frac{1}{2}\right)^2 = \frac{3}{8} - \frac{1}{4} = \frac{3-2}{8} = \frac{1}{8}$

- ACF

a sample (random variable due to fix time in a stochastic process) of $Y(n_1) = \frac{1}{2}(X(n_1) + X(n_1 - 1))$ is the linear combination of two random variables with specific statistical properties.

For any other sample of the process

$Y(n_2) = \frac{1}{2}(X(n_2) + X(n_2 - 1))$ the random variables will have the same identical statistical properties as before. (due to $X(n)$ being stationary)

Since this holds $\forall n$ $Y(n)$ is stationary.

Then $r_y(n_1, n_2)$ will only depend on the sample shift $k = n_1 - n_2$.

$$r_y(k) = E[Y(n+k)Y(n)] = \frac{1}{4} E[(X(n+k) + X(n+k-1))(X(n) + X(n-1))] =$$

$$= \frac{1}{4} E[\underbrace{X(n+k)X(n)}_{r_x(k)} + \underbrace{X(n+k)X(n-1)}_{r_x(k+1)} + \underbrace{X(n+k-1)X(n)}_{r_x(k-1)} + \underbrace{X(n+k-1)X(n-1)}_{r_x(k)}]$$

shift between the two

$$= \frac{1}{4}(2r_x(k) + r_x(k+1) + r_x(k-1)) =$$

$r_x(k)$:
- $k = 0$: $E[X^2(n)] = \frac{1}{2}(1)^2 + \frac{1}{2}(0)^2 = \frac{1}{2}$
- $k \neq 0$: $E[X(n)X(n+k)] \rightarrow$ independent of other samples $= E[X(n)]E[X(n+k)] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

If all arguments of the correlations above are different than $\emptyset$ we have

$$r_y(k) = \frac{1}{4}\left(2\frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4}.$$

The only cases in which one of the arguments will be $\emptyset$ is for $k=0$, $k=1$ and $k=-1$

Hence:
- $r_y(0) = \frac{1}{4}(2r_x(0) + r_x(1) + r_x(-1)) = \frac{1}{4}\left(2\frac{1}{2} + \frac{1}{4} + \frac{1}{4}\right) =$
  $= \frac{1}{4}\left(1 + \frac{1}{2}\right) = \frac{1}{4} + \frac{1}{8}$ //

- $r_y(1) = \frac{1}{4}(2r_x(1) + r_x(0) + r_x(2)) = \frac{1}{4}\left(2 \cdot \frac{1}{4} + \frac{1}{2} + \frac{1}{4}\right) =$
  $= \frac{1}{4}\left(1 + \frac{1}{4}\right) = \frac{1}{4} + \frac{1}{16}$ //

- $r_y(-1) = r_y(1)$ (property of acf. for stationary processes)

$$r_y(k) = \frac{1}{4} + \begin{cases} \frac{1}{8} & \text{for } k=0 \\ \frac{1}{16} & \text{for } k=-1 \text{ or } 1 \\ \emptyset & \text{for the rest} \end{cases}$$

2.6.

$X(t)$ and $Y(t)$ are independent and wide sense stationary stochastic processes. $\Rightarrow E[X(t)Y(t)] \underset{\uparrow}{=} E[X(t)]E[Y(t)]$

independent

$\Rightarrow E[X(t)] = m_X \quad E[Y(t)] = m_Y \quad r_X(t_1, t_2) = r_X(\tau) \quad (\tau = t_1 - t_2)$

$\uparrow$ wide sense stationary $\qquad r_Y(t_1, t_2) = r_Y(\tau)$

- when 2 processes are independent

$Z(t) = X(t) + Y(t)$

$U(t) = X(t) \cdot Y(t)$

($\Rightarrow$ all random variables that come from one of them are independent of all the ones that come from the other)

- For the processes to be wide sense stationary the mean has to be constant and the a.c.f. shall dependent only on the shift.

→ Mean:

$E[Z(t)] = E[X(t) + Y(t)] \rightarrow \{\text{linear}\} = E[X(t)] + E[Y(t)] =$

$= m_X + m_Y$.

$E[U(t)] = E[X(t)Y(t)] \underset{\uparrow}{=} E[X(t)]E[Y(t)] = m_X m_Y$.

independent

For both the mean is constant!

$\longrightarrow$

Next page

→ auto-correlation function :

* assume non-stationarity :

$ACF(Z(t)) = E[Z(t_1)Z(t_2)] = E[(X(t_1)+Y(t_1))(X(t_2)+Y(t_2))]$

$= \left\{ \begin{array}{l} \text{the expectation} \\ \text{is linear} \end{array} \right\} = \underbrace{E[X(t_1)X(t_2)]}_{r_X(t_1,t_2) = r_X(\tau)\ \ (\tau = t_1 - t_2)} + E[X(t_1)Y(t_2)] +$

$+ E[X(t_2)Y(t_1)] + E[Y(t_1)Y(t_2)] \underset{\substack{\uparrow \\ \text{x and y are} \\ \text{w.s. stationary}}}{=} r_X(\tau) + r_Y(\tau) +$

$+ E[X(t_2)]E[Y(t_1)] + E[X(t_1)]E[Y(t_2)] = \left\{ \begin{array}{l} \text{X and Y are} \\ \text{independent} \end{array} \right.$

$\underset{\substack{\uparrow \\ \text{w.s. stationarity} \\ \text{X and Y}}}{=} r_X(\tau) + r_Y(\tau) + 2m_X m_Y$ ■

$ACF(U(t)) = E[U(t_1)U(t_2)] = E[X(t_1)Y(t_1)X(t_2)Y(t_2)] =$

$= \left\{ \begin{array}{l} \text{X and Y} \\ \text{are independent} \end{array} \right\} = E[X(t_1)X(t_2)]E[Y(t_1)Y(t_2)] =$

$= \left\{ \begin{array}{l} \text{X and Y} \\ \text{are w.s. stationary} \end{array} \right\} = r_X(\tau)\, r_Y(\tau)$. ■

$\boxed{2.11}$

$A(k)$: Can not be an ACF of a wide-sense stationary process since it has points that have the same modulus as $A(0)$ but is not periodic (it ends).

$B(k)$: $B(0)$ is the maximum value in modulus
it's symmetric with k
It CAN be an ACF.

$C(k)$: it cannot since $C(0) < C(1)$
$C(0) < (2)$

$D(k)$: it cannot since $D(0) < |D(2)| = |D(-2)|$