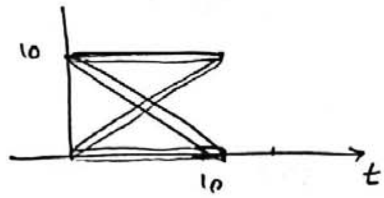
1.1) a) $Z(t)=t \qquad x=0 \quad y=1$

$Z(t)=0 \qquad x=0 \quad y=0$

$Z(t)=10-t \qquad x=1 \quad y=0$

$Z(t)=10 \qquad x=1 \quad y=1$

$\Longrightarrow$



b) $r_Z(t_1,t_2)=E[Z(t_1)Z(t_2)]=E\Big[\big((10-t_1)x+t_1 y\big)\big((10-t_2)x+t_2 y\big)\Big]$

$=(10-t_1)(10-t_2)\underbrace{E[x^2]}_{1/2}+\big((10-t_1)t_2+(10-t_2)t_1\big)\underbrace{E[xy]}_{1/4}+t_1t_2\underbrace{E[y^2]}_{1/2}$

$=50-\frac{5}{2}t_1-\frac{5}{2}t_2+\frac{t_1t_2}{2}$

★ $E[x^2]=0^2\times P(x=0)+1^2\times P(x=1)=\frac{1}{2}$

$E[y^2]=E[x^2]$

$E[xy]=0\times\big(P(x=0,y=0)+P(x=0,y=1)+P(x=1,y=0)\big)+1\times\big(P(x=1,y=1)\big)$

$=P(x=1,y=1)=P(x=1)\,P(y=1)=\frac{1}{4}$

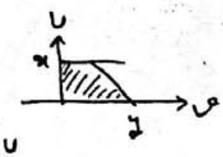
independent

OR, $E[xy]=E[x]E(y)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

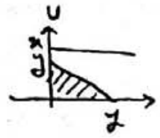
1.12) a) Method ① $f_{xy}=f_x f_{y|x}=f_U(x)\,f_{y|x}(y)=e^{-x}\underbrace{e^{-y+x}}_{x<y}=e^{-y} \quad 0<x<y$

Method ② General $F_{xy}(x,y)=P(x\leq x, y<y)$

Ⅰ $x\leq y \quad U=x \quad V=y-U$



Ⅱ $x\geq y$



Ⅰ $\rightarrow F_{xy}=\int_0^x\int_0^{y-u}e^{-u}e^{-v}\,dv\,du$

$=\int_0^x e^{-u}(1-e^{u-y})\,du=1-e^{-x}-xe^{-y}$

Ⅱ $\rightarrow F_{xy}=\int_0^y\int_0^{y-u}e^{-u}e^{-v}\,dv\,du=1-e^{-y}-ye^{-y}$

$f_{xy}=\begin{cases}\dfrac{\partial^2 F_{xy}}{\partial x\,\partial y}=e^{-y} & 0<x<y\\ 0 & \text{o.w.}\end{cases}$

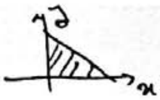
Method ③ More general

Jacobian

b) $f_{x|y}=\begin{cases}\frac{1}{y} & 0<x<y\\ 0 & \text{o.w}\end{cases}$

c) $m_{x|y}=E\{x|y=y\}=\int_0^y x\frac{1}{y}\,dx=\frac{y}{2}$

1.14) $E[X] = \int x f(x)\,dx$ $\qquad f_{xy}(x,y) = 2 \quad 0 \le x \le 1, \; 0 \le y \le 1-x$



$$\Rightarrow f_x(x) = \int_0^{1-x} 2 = 2(1-x)$$

$$\Rightarrow E[X] = \int_0^1 2x(1-x)\,dx = x^2 - \frac{2}{3}x^3 \Big|_0^1 = \frac{1}{3}$$

Due to symmetry $\rightarrow E[y] = E[X]$

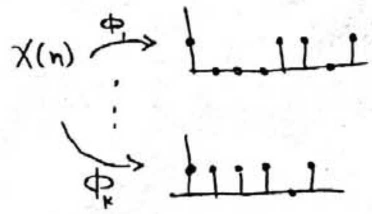
$$r(x,y) = E[xy] = \int_0^1 \int_0^{1-x} 2xy \; dx\,dy = \int_0^1 xy^2 \Big|_0^{1-x} dx = \int_0^1 x(1-x)^2\,dx = \frac{1}{12}$$

$$Cov = r(x,y) - m_x m_y = \frac{1}{12} - \frac{1}{3} \times \frac{1}{3} = \frac{-1}{36}$$

$$E[x^2] = \int_0^1 2x^2(1-x) = \frac{1}{6} \Rightarrow Var(x) = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

$$\rho = \frac{-1/36}{1/18} = -1/2$$

2.2)



in each time instance 'n' a realization of Process X(n) has probability density function of

$$P(X(n)=0) = P(X(n)=1) = \frac{1}{2}$$

and independent of other time instances.

$$E[X(n)] = \frac{1}{2} \times 1 + 0 = \frac{1}{2}$$

$$E[X^2(n)] = \frac{1}{2} \times 1 = \frac{1}{2}$$

$$E[X(n+k)X(n)] = E[X(n)]\underbrace{E[X(n+k)]}_{=E[X(n)]} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \qquad k \neq 0$$

stationary

$\boxed{\text{How to make not stationary?}}$

a) $m_y = E[y(n)] = \frac{1}{2} E[X(n)] + \frac{1}{2} E[X(n-1)] = \frac{1}{2}$

b) $P_y = E[y^2(n)] = E\left[\frac{1}{4}\left(X^2(n) + X^2(n-1) + 2X(n)X(n-1)\right)\right] = \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2} + \frac{1}{2} \underbrace{r_X(1)}_{=\frac{1}{4}} = \frac{3}{8}$

c) $\sigma_y^2 = 3/8 - \frac{1}{4} = 1/8$

d) $r_y(k) = E[y(n+k)y(n)] = E\left[\frac{1}{2}\left(X(n+k) + X(n+k-1)\right) \times \frac{1}{2}\left(X(n) + X(n-1)\right)\right]$

$$= \frac{1}{4}\left(r_X(k) + r_X(k+1) + r_X(k-1) + r_X(k)\right) = \begin{cases} \frac{3}{8} & k=0 \\ \frac{5}{16} & k=1,-1 \\ \frac{1}{4} & \text{o.w.} \end{cases}$$

2.6) $E[z(t)] = m_x + m_y \rightarrow$ independent of time

$E[u(t)] = m_x m_y$

$$E[z(t+\tau)z(t)] = E[x(t+\tau)x(t) + x(t+\tau)y(t) + x(t)y(t+\tau) + y(t+\tau)y(t)]$$

$$= r_x(\tau) + r_y(\tau) + 2m_x m_y \rightarrow \text{function of } \tau \text{ only}$$

2.11)

A(k)

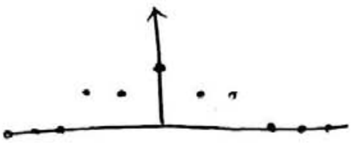
$A(0) \geq 0$ ✓

$\forall k \; A(k) = A(-k)$ ✓

$\forall k \; A(0) \geq |A(k)|$ ✓

$\left.\begin{matrix} A(0) = A(2) = A(-2) \\ A(0) \neq A(4) \end{matrix}\right\}$ X

B(k)



$B(0) \geq 0$ ✓

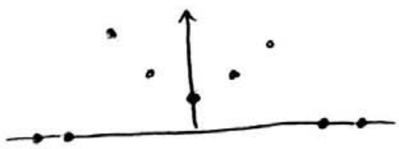
$\forall k \; B(k) = B(-k)$ ✓

$\forall k \; B(0) \geq |B(k)|$ ✓

$B(0) \neq B(k) \; \forall k \neq 0$ ✓

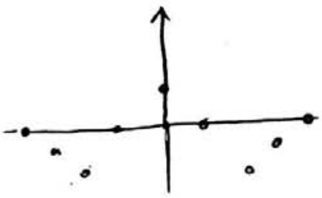
WSS

C(k)



$C(0) < C(1)$ X

D(k)



$C(0) < |C(2)|$ X