- A <u>weak k-composition of n</u> is a sequence $(x_1, x_2, \dots, x_k)$ of nonnegative integers s.t.

$$x_1 + x_2 + \dots + x_k = n$$

Let $W_{n,k} = \{\text{weak } k\text{-comp. of } n\}$

<u>Fact</u>: $|W_{n,k}| = \binom{n+k-1}{k-1}$

<u>Proof</u>: Define $\varphi : W_{n,k} \to C_{n+k,k}$ by $\varphi(x_1, \dots, x_k) = (x_1+1, \dots, x_k+1)$. φ is a bijection. $\square$

- A <u>multiset</u> is a set with "repetitions" allowed: $\{1,1,1,4,6,6,8\}$.

Let $\left(\!\binom{n}{k}\!\right) := |\{k\text{-multisets of } [n]\}|$

$$\left(\!\binom{[n]}{k}\!\right)$$

<u>Fact</u>: $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}$

Define $\varphi : \left(\!\binom{[n]}{k}\!\right) \to W_{k,n}$ by

$\varphi(S) = (x_1, \dots, x_n)$ where

$x_j = \#\ j\text{'s in } S$

Hence $x_1 + \dots + x_n = k$.

<u>Ex</u>: $S = \{1,1,1,4,6,6,8\} \in \left(\!\binom{[9]}{7}\!\right)$

$\varphi(S) = (3,0,0,1,0,2,0,1,0)$.

- Recall that if $\alpha \in \mathbb{R}$ is any real number, then

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} \quad \text{and}$$

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

- Fact $\quad (1-x)^{-n} = \sum_{k \geq 0} \left(\!\binom{n}{k}\!\right) x^k \quad$ and hence

$$(1+x)^{n} = \sum_{k \geq 0} \left(\!\binom{n}{k}\!\right)(-1)^k x^k \quad \text{and hence}$$

$$\binom{-n}{k} = \left(\!\binom{n}{k}\!\right)(-1)^k \quad \text{(Reciprocity)}.$$

<u>Proof</u>: $\quad (1-x_1)^{-1} \dots (1-x_n)^{-1} =$

$$= (1+x_1+x_1^2+\dots)(1+x_2+x_2^2+\dots)\dots(1+x_n+x_n^2+\dots)$$

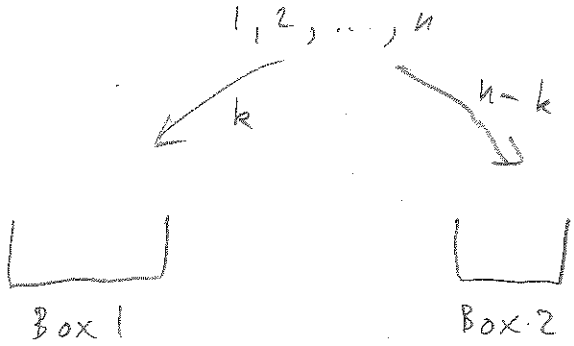
$$= \sum_{a_1, a_2, \dots, a_n \geq 0} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$$

Set $x_j = x$ for all j !

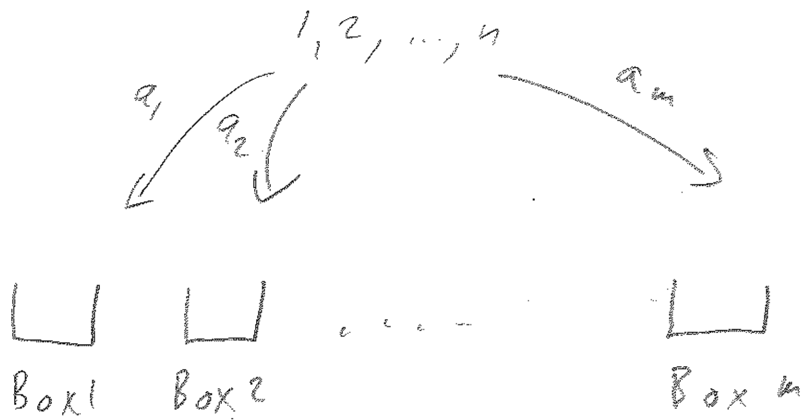
$$(1-x)^{-n} = \sum_{k} |W_{k,n}| x^k = \sum_k \left(\!\binom{n}{k}\!\right) x^k$$

$\square$

- Note that we may interpret $\binom{n}{k}$ as the number of ways to put the numbers $1, 2, \ldots, n$ into 2 boxes such that the first box gets exactly k integers!



- What about m boxes?
- Suppose $a_1, \ldots, a_m$ are nonnegative integers s.t. $a_1 + a_2 + \cdots + a_m = n$. Define the <u>multinomial number</u> $\binom{n}{a_1, a_2, \ldots, a_m}$ to be the number of ways to put $1, 2, \ldots, n$ into m boxes so that box i gets a_i integers;



<u>Formula</u>: $$\binom{n}{a_1, \ldots, a_m} = \frac{n!}{a_1! a_2! \cdots a_m!}$$

<u>Proof</u>: Same proof as for binomials.

Multinomial theorem:

$$(x_1+x_2+\dots+x_m)^n = \sum_{a_1+a_2+\dots+a_m=n} \binom{n}{a_1,a_2,\dots,a_m} x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m}$$

Proof: Exercise!

- If M is a multiset we let $\mathfrak{S}(M)$ denote the set of "distinct" permutations of M:
- If $M=\{1,1,2\}$, then $\mathfrak{S}(M)=\{112, 121, 211\}$

Theorem: Suppose M is a multiset of $[k]$ with exactly a_i copies of i for each $i\in[k]$. Then

$$|\mathfrak{S}(M)| = \binom{n}{a_1,\dots,a_k},$$

where $a_1+a_2+\dots+a_k=n$.

Proof: Given $\pi\in\mathfrak{S}(M)$ we want to bijectively assign a placement of $1,2,\dots,n$ into k boxes:

If position j in π is occupied by integer r, then j goes in the r'th box.

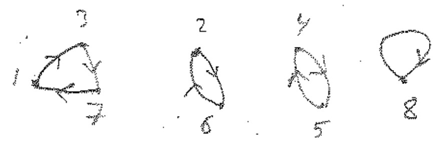
$$\begin{array}{cccccccc} 4&2&1&2&2&4&1&1\\ {\scriptstyle 1}&{\scriptstyle 2}&{\scriptstyle 3}&{\scriptstyle 4}&{\scriptstyle 5}&{\scriptstyle 6}&{\scriptstyle 7}&{\scriptstyle 8}\end{array} \longrightarrow \begin{array}{cccc} 8 & 5 & & \\ 7 & 4 & & 6 \\ \boxed{3} & \boxed{2} & \boxed{\emptyset} & \boxed{1} \\ 1 & 2 & 3 & 4 \end{array}$$

This is a bijection. $\square$

1.3 Cycles and inversions.

- $\mathfrak{S}(S) := \{\text{permutations of } S\}$
- Group structure (composition)
- Cycle structure:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 7 & 5 & 4 & 2 & 1 & 8 \end{pmatrix}$$



$$\pi = \underbrace{(137)(26)(45)(8)}_{\text{disjoint product of cycles (not unique)}} = (54)(371)(8)(26)$$

Def: Standard representation

(a) Each cycle is written with its largest element first

(b) The cycles are written in increasing order of their largest element.

$$\pi = (54)(62)(713)(8)$$

Def: Define a map $\wedge : \mathfrak{S}_n \to \mathfrak{S}_n$ by removing the brackets in standard representation:
$\hat{\pi} = 54\,62\,713\,8$

Clearly $\wedge$ is a bijection.

Def: A left-to-right maximum in a permutation $\sigma = a_1 a_2 \ldots a_n$ is an a_i s.t. $a_j < a_i$ for every $j < i$.

$lrm(\sigma) := \#$ l-r-max.

$lrm(\underline{5}4\underline{6}2\underline{7}13\underline{8}) = 4$

Clearly a_i starts a cycle in $\pi \Leftrightarrow a_i$ is l-r-max in $\hat{\pi}$

$\therefore \quad lrm(\hat{\pi}) = c(\pi) := \#$ cycles in π

The *cycle type* of π is the vector $\bar{c}(\pi) = (c_1(\pi), c_2(\pi), \dots, c_n(\pi))$
where $c_i(\pi) = \#$ cycles of length i in π.

$\bar{c}(36754218) = (1, 2, 1, 0, 0, 0, 0)$. Hence $n = \sum i c_i$
$c(\pi) = \sum c_i$

Prop: Let $\bar{c} = (c_1, \dots, c_n)$ be fixed.

$$\Big|\underbrace{\{\pi \in \mathfrak{S}_n : \bar{c}(\pi) = \bar{c}\}}_{\mathfrak{S}_{\bar{c}}}\Big| = \frac{n!}{1^{c_1} c_1!\, 2^{c_2} c_2! \cdots n^{c_n} c_n!}$$

Proof: Define $\Phi : \mathfrak{S}_n \to \mathfrak{S}_{\bar{c}}$ as follows.

If $\tau \in \mathfrak{S}_n$ parenthesize τ so that the
first c_1 cycles have length 1
next c_2 cycles have length 2

Ex: $\tau = 427619583$, $\bar{c} = (1, 2, 0, 1, 0, 0, \dots)$

$\Phi(\tau) = (4)(27)(61)(9583)$

For $\sigma \in \mathfrak{S}_{\bar{c}}$, compute $|\Phi^{-1}(\sigma)| = |\{\tau \in \mathfrak{S}_n : \Phi(\tau) = \sigma\}|$

Divide τ into "blocks" according to $\bar{c}$.

$\tau = 4 | 27 | 61 | 9583$

- There are $c_i!$ ways of deciding which order the blocks of length i should come.
- For each cycle of length i we decide which of the i elements should be the first element in its block.

$$\underbrace{i \cdot i \cdots i}_{c_i} = i^{c_i} \text{ choices}$$

- These choices are independent.

$\therefore \quad |\Phi^{-1}(\sigma)| = \prod_i c_i!\, i^{c_i}$

$$\therefore \quad n! = \sum_{\sigma \in \mathfrak{S}_c} |\Phi^{-1}(\sigma)| = \sum_{\sigma \in \mathfrak{S}_c} \prod_i c_i!\, i^{c_i}$$

$$= |\mathfrak{S}_c| \prod_i c_i!\, i^{c_i} \qquad \square$$

Let $Z_n = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} t_1^{c_1(\pi)} \cdots t_n^{c_n(\pi)}$, and let

$$Z = \sum_{n=0}^{\infty} Z_n x^n =$$

$$= \sum_{c_1=0}^{\infty} \sum_{c_2=0}^{\infty} \cdots \; x^{1\cdot c_1 + 2\cdot c_2 + \cdots} \frac{t_1^{c_1}}{c_1!\, 1^{c_1}} \cdot \frac{t_2^{c_2}}{c_2!\, 2^{c_2}} \cdots$$

$$= \sum_{c_1=0}^{\infty} \sum_{c_2=0}^{\infty} \cdots \frac{\left(\frac{x^1 t_1}{1}\right)^{c_1}}{c_1!} \cdot \frac{\left(\frac{x^2 t_2}{2}\right)^{c_2}}{c_2!} \cdot \frac{\left(\frac{x^3 t_3}{3}\right)^{c_3}}{c_3!} \cdots$$

$$= e^{x t_1} \cdot e^{x^2 t_2/2} \cdot e^{x^3 t_3/3} \cdots$$

$$= \exp\left(\sum_{k \geq 1} \frac{t_k x^k}{k} \right).$$

Example: π is an involution if $\pi^2 = id \iff$ $c_3(\pi) = c_4(\pi) = \ldots = 0$

Let $I_n = \#$ involutions in $\mathfrak{S}_n$, then

$$\sum_{n=0}^{\infty} I_n \frac{x^n}{n!} = \exp\left(x + \frac{x^2}{2}\right)$$