Exercise 9 chapter 2.

This is included in your calculus in 1 variable course. But it is important so I want to supply a proof:

1) It is enough to prove that increasing sequences bdd from above converges since if a_j is decreasing and bounded below then $-a_j$ will be increasing & bdd above (and if $-a_j$ converges then a_j converges.).

2) ~~Def.~~ Let a_j be increasing and bdd below. Define $A = \{a_j ; j \in \mathbb{N}\}$ then $A \neq \emptyset$ and A is bdd above.

Thus $l.u.b(A) = a$ exists.

3) For any $\varepsilon > 0$ $a - \varepsilon$ is not an upper bound of A (since $a - \varepsilon < a = l.u.b(A)$).

Therefore $\exists a_J \in A$ s.t. $a - \varepsilon < a_J$.

4) We claim that $j > J \Rightarrow |a - a_j| < \varepsilon$. This is clear since if $j > J$

$$a - \varepsilon < a_J \leq \left\{\begin{matrix} a_j \\ \text{increasing} \end{matrix}\right\} \leq a_j \leq \left\{\begin{matrix} a \text{ is} \\ \text{upper bound} \end{matrix}\right\} \leq a < a + \varepsilon$$

$\Rightarrow j > J \Rightarrow |a - a_j| < \varepsilon$, Thus $a_j \to a$. ∎

Ex 14 chapter 2.

a) We need to show that if $\cancel{x_j} \to x_0$ in M then $f(\cancel{x_j}) \to f(x_0)$ in N.

But this is clear since

$x_j \to x_0$ in M means that $d(x_j, x_0) \to 0$

but, since f is an isometry,

$$d(f(x_j), f(x_0)) = d(x_j, x_0) \to 0$$

which means that $f(x_j) \to f(x_0)$.

$\boxtimes$

b) We need to show

1) f is a bijection
2) f is continuous
3) f^{-1} — " —

But 1) follows from the fact that f is an isometry (which includes the assumption that f is a bijection). We proved 2) in part a).

Therefore we need to show that f^{-1} is continuous. We will show that f^{-1} is an isometry and then the result follows from part a).

To that end we let $y_1, y_2 \in N$ be arbitrary points. Since f is a bijection there exsists points s.t. $f(x_1) = y_1$ and $f(x_2) = y_2$ and

$x_1 = f^{-1}(y_1)$ and $\cancel{x_2} = f^{-1}(y_2)$

Therefore

$$d(y_1,y_2) = d(\underbrace{f(x_1)}_{=y_1},\underbrace{f(x_2)}_{=y_2}) = \{f \text{ isometry}\} = d(x_1,x_2) = d(f^{-1}(y_1),f^{-1}(y_2))$$

We have thus shown that

$$d(y_1,y_2) = d(f^{-1}(y_1),f^{-1}(y_2)).$$ ~~which~~ impl

This together with the fact that f^{-1} is a bijection (since f is a bijection) implies that f^{-1} is an isometry and thus, by part a), continuous. We have shown 1), 2), 3) and are therefore done.

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c) Assume that $f: [0,1] \longmapsto [0,2]$ is an isometry

a) Since f is a bijection it is onto and we may find $x_1, x_2 \in [0,1]$ such that $f(x_1) = 0$ and $f(x_2) = 2$.

Since f is an isometry it follows that

$$d(x_1,x_2) = d(f(x_1),f(x_2)) = d(0,2) = 2. \qquad 1$$

but also, since $x_1,x_2 \in [0,1]$ it follows that $d(x_1,x_2) \leq 1$. 2

By 1 and 2 $2 \leq 1$ which is clearly a contradiction. Thus no isometry from $[0,1]$ to $[0,2]$ exsists.

$\boxtimes$

Ex 26: Chapter 2.

First we prove that if U is open then none of its points is a limit of the complement U^c. Assume the contrary, that $x_0 \in U$ and $x_j \to x_0$ for some sequence $x_j \in U^c$. But since $x_0 \in U$ and U is open there exsist an $r>0$ s.t. $M_r x_0 \subset U$, thus

$$x_j \in U^c \Rightarrow d(x_0, x_j) \geq r.$$

But then $d(x_0, x_j) \not\to 0$.

Next assume that none of the points in U is a limit point of U^c, we want to show that U is open. We need to show that if $x_0 \in U$ then $\exists r>0$ s.t. $M_r x_0 \subset U$.

We will show that we can choose $r = \frac{1}{j}$ for some $j \in \mathbb{N}$. If not, then

$M_{1/j}\, x_0 \not\subset U$ for any j. But this means

that $\exists x_j \in M_{1/j}(x_0) \cap U^c$ for every $j \in \mathbb{N}$.

But then $d(x_0, x_j) < \frac{1}{j} \to 0$ and therefore

$x_j \to x_0$ contrary to our initial assumption. It follows that $M_{1/j}\, x_0 \subset U$ and the openness of U follows.