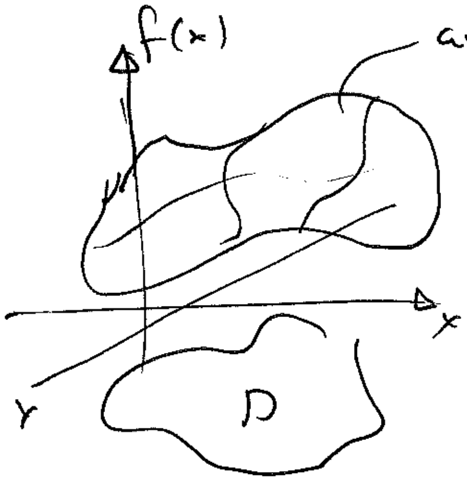
Lecture 1: Remember that the area of the graph $(x, f(x))$, $x \in D$ and $f \in C^1(D)$ is given by

$$\int_D \sqrt{1+|\nabla f|^2}\, dx.$$ ~~Thus if we have a soap film~~



area of graph $= \int_D \sqrt{1+|\nabla f|^2}\, dx.$

Experiments show that a soap bubble will take the shape that has the least area among all possible graphs that ~~span the same region~~ has the same values on ∂D.

Thus the soap bubble will ~~take~~ be described by the function that minimizes $\int_D \sqrt{1+|\nabla f|^2}\, dx$

among all functions g s.t. $g = f$ on ∂D.

Now if $|\nabla f|^2$ is small a Taylor expansion $\left(\sqrt{1+t} \approx 1 + \frac{1}{2} t + O(t^2)\right)$ shows that

$$\int_D \sqrt{1+|\nabla f|^2}\, dx \approx |D| + \underbrace{\frac{1}{2}\int_D |\nabla f|^2\, dx}_{\text{simpler}}.$$

Question: Can we always find a $u: D \to \mathbb{R}$ such that $u = f$ on ∂D and $\int_D |\nabla u|^2 dx \leq \int_D |\nabla v|^2 dx$ among all $v = f$ on ∂D?

Discussion?

How do we approach a new problem?
Two ways. 1) Make it simpler - maybe we will understand something new by solving the simpler problem.
2) Look at similar problems, already solved.

Thm: Let $f : [a,b] \to \mathbb{R}$ be lower semicontinuous ($\lim_{x \to x^0} f(x) \geq f(x_0)$ for all x_0) and bounded from below. Then there exists an $x_0 \in [a,b]$ s.t. $f(x_0) \leq f(x)$ for all $x \in [a,b]$.

Proof: Step 1 $\inf_{x \in [a,b]} f(x) = m$ exists.

By ~~definition~~ assumption $V = \{f(x) ; x \in [a,b]\}$ is non-empty and bounded from below so g.l.b.$(V) = m$ exists

Step 2: $\exists x^j$ s.t. $f(x^j) \to m$.

Since $m + \frac{1}{j}$ is not a lower bound of V $\exists x^j \in V$ s.t. $f(x^j) < m + \frac{1}{j}$ but m is a lower bound so $m < f(x^j) < m + \frac{1}{j} \to m$.

Step 3: $\exists$ subsequence $x^{j_k} \to x^0 \in [a,b]$.

Proof: i) Bolzano-Weierstrass: Every bdd sequence (such as x^j, $a \le x^j \le b$) has a convergent subsequence in

ii) $[a,b]$ is closed so $\lim_{k\to\infty} x^{j_k} \in [a,b]$

step 4 $m \le f(x^0) \le \{f \text{ l.s.c}\} \le \lim_{k\to\infty} f(x^{j_k}) = m$

m lower bound

$\Rightarrow f(x^0) = m = \min U$.

Can we use this argument for minimizing $\int_D |\nabla u|^2 dx$?

Step 1: Same $\inf_K \int_D |\nabla u|^2 dx = m$

step 2: Same Over what? $\exists u^j \in K$ $\int_D |\nabla u^j|^2 \to m$

step 3: Need Bolzano-Weierstrass!

But ~~Bolz~~ B-W depends on the set ($[a,b] \in \mathbb{R}$ vs. $[a,b] \subset \mathbb{Q}$).

What is the natural space to minimize over? Can we get boundedness, well if $\int_D |\nabla u^j|^2 \to m$ then $\int_D |\nabla u^j|^2 dx \le m+1$ for j large.

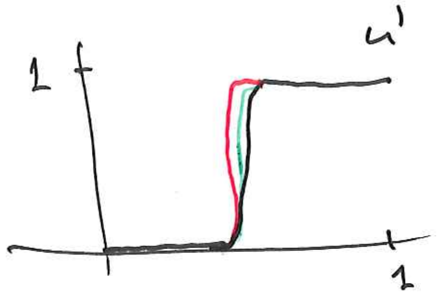
Thm (What we want): Assume

$\int_D |\nabla u^j|^2 \leq M$ then $\exists$ subsequence $\underbrace{u^{j_k} \to u^0}_{\text{(what sense?)}}$.

NOT TRUE $\int_D |\nabla u^j - \nabla u^0|^2 dx \to 0$?

What does it mean for a function to have $\int_D |\nabla u^j|^2 \leq M$? Clearly if $u \in C^1(\bar{D})$ then $\int_D |\nabla u^j|^2 \leq M$ is well defined, but $C^1(\bar{D})$ is certainly not closed under the norm $\int_D |\nabla u|^2$.

Ex



Then $\int_0^1 |\nabla u^j|^2 \leq 1$ but $u^j \not\to u^0 \in C^1$.

So in order to have some set of functions ~~s.t.~~ closed under the $\int_D |\nabla(u^j - u^0)|^2$. convergence we need to enlarge the ~~space~~ concept of derivatives. For instance Darboux thm

Thm: If $f: [a,b]$ is differentiable at every point (not necessarily f' continuous) then for every $f'(a) < r < f'(b)$ there exists a $c \in (a,b)$ s.t. $f'(c) = r$

Proof: $f(x)-rx$ differentiable, thus continuous, on $[a,b]$

$\Rightarrow$ $f(x)-rx$ has minimum (previous argument) at some point c

$\Rightarrow$ c minimum implies $D(f(x)-r(x))=0$ at $x=c$

$\Rightarrow$ $f'(c)=r$ but $f'(a)<r<f'(b)$

so $c\neq a$ & $c\neq b$ $\Rightarrow$ $c\in(a,b)$ & $f'(c)=0$.

Since the only possible limit in the example is

$$Du^\circ(x)=\begin{cases}1 & x>\frac{1}{2}\\ \text{one value} & \\ 0 & x<\frac{1}{2}\end{cases}$$

the limit cannot be differentiable in the classical sense $\Rightarrow$ We need to extend the concept of the derivative to solve the problem.

Next problem, is K_E closed

very difficult to answer since we need to first understand the new integral and derivative, that we need to satisfy some sort of B-W thm,...

...

Step 4. Also need l.s.c. for $\int_D |Du|^2$ with respect to the kind of convergence that gives B-W

The course:

We will try to solve the variational problem, mostly because that will force us to cover much mathematical analysis. We will in particular

1) Discuss the Lebesgue integral (Good convergence)
You should have seen this.
2) Prove that every bounded sequence u^j, in the sense $\int_D |u^j|^2 dx \leq M$, has a convergent subsequence (if we interpret convergence as weak convergence).
Some functional analysis.
3) Extend the concept of differentiability to weak derivatives (Sobolev spaces, u s.t. $\int |\nabla u|^2 < \infty$)
4) Discuss Traces (boundary values) and compactness in Sobolev spaces

10 Lectures.

~~5)~~ + 3 more lectures.

Books: Rudin or Fitzpatrick.

We will not really need the generality – lectures will often be at a lower level but you need a book.

2 Homework ~~exercises~~ assignments

1 Oral exam.

Today: Riemann integral.

Def: Riemann integral: $f:[a,b]\longrightarrow\mathbb{R}$ is Riemann integrable

if: $\varepsilon>0$ $\exists$ $\exists$ partition $\underbrace{a=x_0<x_1<\dots<x_n=b}_{\text{partition } P}$ s.t.

$$\sum_{k=1}^{n} \underbrace{\operatorname*{osc}_{(x_{k-1},x_k)} f}\,(x_k-x_{k-1})<\varepsilon$$

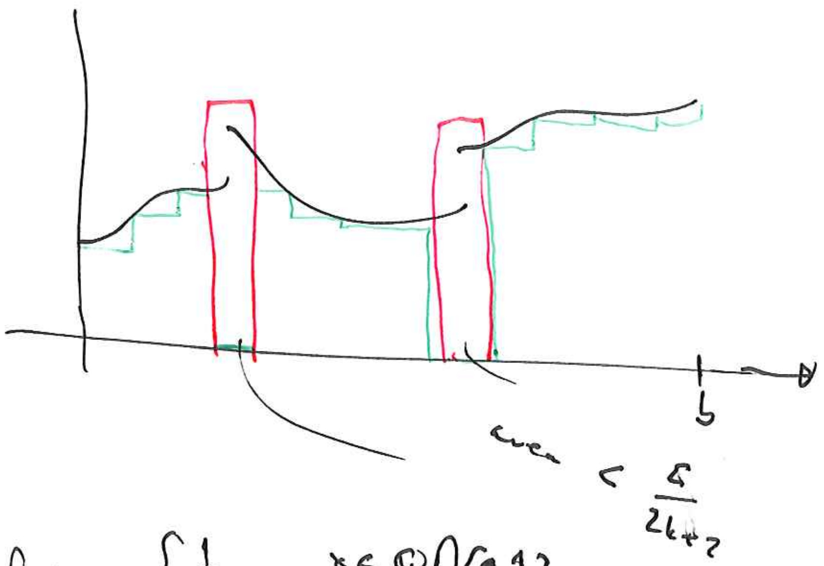
$$\boxed{\operatorname*{osc}_{(x_{k-1},x_k)} f = M_k-m_k=\sup_{(x_{k-1},x_k)} f-\inf_{(x_{k-1},x_k)} f}$$

Define

$$\int_a^b f(x)\,dx=\sup_P \sum_{k=1}^{n} m_k(x_k-x_{k-1})$$

where $\sup_P$ is the supremum over all partitions P. and all $n\in\mathbb{N}$.

Ex: f discont at finitely many points say k points and f bounded



Ex: $f(x)=\begin{cases}1 & x\in\mathbb{Q}\cap[0,1]\\ 0 & x\in[0,1]\setminus\mathbb{Q}\end{cases}$ then

f is zero on a dense set

so $M_k-m_k=1$ for all $(x_{k-1};x_k)$

f not integrable.

Question: Which functions are Riemann integrable?

From Ex 1 and Ex 2 we may guess:

f is Rie-Int if the set where f is discont is "small".

How do we measure smallness?

Def: We say that a set $S \subset \mathbb{R}$ has measure zero if for every $\varepsilon > 0$ there exist a countable cover: (a_i, b_i) s.t. $S \subset \bigcup_{\substack{i=1 \\ i \in \mathbb{N}}}^{\infty} (a_i, b_i)$

s.t. $\sum_{i=1}^{\infty} (b_i - a_i) < \varepsilon$

Thm: Let $f: [a,b] \to \mathbb{R}$ be bounded. Then f is Rie-Int iff the set where f is discontinuous has measure zero.

Remark: Quite fantastic Thm, besides being very powerful, we can use ~~the~~ a countable covering (measure zero def) in order to

$\not{b}$ — ⌒

Proof: Notice that

$$D = \{x \in [a,b];\ f \text{ discont at } x\} = \bigcup_{k=1}^{\infty} D_k$$

where $D_k = \{x;\ \lim_{r\to 0^+} \underset{[x-r,x+r]}{\operatorname{osc}} f \geq \frac{1}{k}\}$.

So if each D_k has measure zero then D will have measure zero.

Proof: Cover D_k by $(a_1^k, b_1^k), (a_2^k, b_2^k) \ldots$

s.t. $\sum_{j=1}^{\infty} (b_j^k - a_j^k) \leq \frac{\varepsilon}{2^k}$.

Then (a_j^k, b_j^k) is a countable cover of $\bigcup_{k=1}^{\infty} D_k = D$ and

$$\sum_{j,k\geq 1}^{\infty} (b_j^k - a_j^k) = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} \leq \varepsilon .$$

~~Step 1) f integrable $\Rightarrow$ D_k zero measure.~~

~~Since f is integrable~~

Claim: D_k is closed

It is enough to show that if $x_j \to x^0$, $x_j \in D_k$ then $x_0 \in D_k$. Assume that $x_0 \notin D_k$ then $\exists r > 0$ s.t.

$$\frac{1}{k} > \underset{[x_0-r,x_0+r]}{\operatorname{osc}} f \geq \underset{[x_j - r_{x_j}, x_j + r_{x_j}]}{\operatorname{osc}} f \geq \frac{1}{k} \quad \text{since } x_j \in D_k .$$

If $|x_j - x_0| < \frac{r}{2}$ and $r_{x_j} < r$

contradiction!

<u>Claim:</u> f integrable $\Rightarrow$ D_k has measure zero.

proof Choose ε/k in the def of Riemann integral.

<u>claim:</u> D_k measure zero for all k $\Rightarrow$ f integrable

Choose $\varepsilon > 0$ we need to show that

$\exists$ partition $a = x_0 < x_1 < \dots < x_n = b$ s.t.

$$\sum_{k=1}^{n} (x_k - x_{k-1}) \operatorname*{osc}_{(x_k, x_{k-1})} f .$$ To that end choose

$$\frac{1}{k} < \frac{\varepsilon}{4(b-a)M} .$$

Since D_k has measure zero we may cover D_k by countably many intervals (a_i, b_i) s.t. $\sum (b_i - a_i) < \varepsilon/4M$ ($M = \sup(f)$).

~~Then~~ But since D_k is closed and bdd. (thus compact) we may cover D_k by finitely many intervals.

Also for $x \in [a,b] \setminus \bigcup_k (a_i, b_i)$ (also compact since $\bigcup (a_i, b_i)$ is open)

There is an $r_x > 0$ s.t.

$$\operatorname*{osc}_{(x - r_x,\, x + r_x)} f < \frac{1}{k} < \frac{\varepsilon}{4(b-a)}$$

Then $\bigcup (x - r_x, x + r_x)$ is an open cover for $[a,b] \setminus \bigcup (a_i, b_i)$ compact so we have a finite subcover say $(x - r_x, x + r_x)$ $x \in J$ = finite set.

Thus $$[a,b] \subset \underbrace{\Big(\bigcup_{x \in J} (x - r_x, x + r_x) \Big) \bigcup \bigcup (a_i, b_i)}_{\text{finite cover.}}$$

So we may form a partition consisting of all $\underbrace{x \pm r_x \text{ for } x \in J \text{ and } a_i \text{ and } b_i}_{\text{finitely many}}$

Say $a = x_0 < x_1 < \dots < x_n = b$. Now

$$\sum_{k=1}^{n} (x_k - x_{k-1}) \operatorname{osc} f = \sum_{\substack{(x_{k-1}, x_k) \subset (a_i, b_i) \\ \text{for some } i}} (x_k - x_{k-1}) \underbrace{\operatorname{osc} f}_{< 2M}$$

$$+ \sum_{(x_{k-1}, x_k) \in (x - r_x, x + r_x)} (x_k - x_{k-1}) \underbrace{\operatorname{osc} f}_{< \frac{\varepsilon}{4(b-a)}} < \frac{\varepsilon}{4} + 2M \underbrace{\sum_{\substack{(x_{k-1}, x_k) \\ \subset (a_i, b_i)}} (x_k - x_{k-1})}_{< \frac{\varepsilon}{4M}}$$

$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon.$$