L 2

Last lecture we considered the following function

$$f(x) = \begin{cases} 1 & \mathbb{Q} \cap [0,1] \\ 0 & [0,1] \setminus \mathbb{Q} \end{cases}$$

and concluded that f was not integrable in the Riemann sense.

Who cares? It is a very special (and artificial) function created out of malice to show that the Riemann integral is lacking.

However, consider

Example: Let q_j be a numbering of all points in $[0,1] \cap \mathbb{Q}$ (since $\mathbb{Q}$ is countable we may find such sequence q_j) and define

$$f_k(x) = \begin{cases} 1 & \text{if } x = q_j \text{ for } j < k \\ 0 & \text{else.} \end{cases}$$

Then $\int_0^1 f_k(x)\,dx = 0$ (Riemann integral since f_k is discont at the zero set $\{q_j ; j<k\}$)

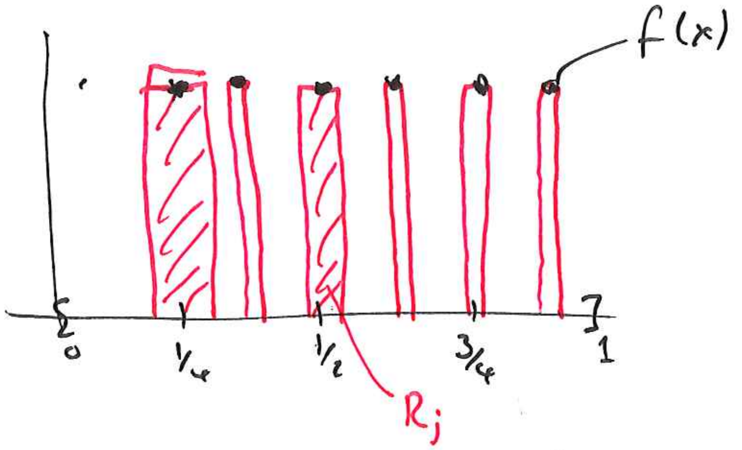
and $f_k(x) \nearrow f(x)$ (pointwise convergence), f not Rie-Int!!

Thus, the Riemann integral does not behave well under convergence, not even the nice monotone and pointwise kind of convergence.

This is a problem!

Another "problem" is that we can, heuristically, integrate $f(x)$ by hand. That is, we can find a "reasonable" answer to the question "What is the area under $f(x)$, $0 \leq x \leq 1$."

Consider q_j as in the example. Then, for any $\varepsilon > 0$, the area under the graph of $f(x)$ should be less than the area of the rectangles $R_j = \left(q_j - \frac{\varepsilon}{2^j}, q_j + \frac{\varepsilon}{2^j}\right) \times [0,1]$ since the union, over all j, of these rectangles contain the area under the graph.



But the area of R_j is $\frac{\varepsilon}{2^j}$ so the area of the union "should be"

$$\sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$

Thus the area under f should be $< \varepsilon$ for all $\varepsilon > 0$. The area under f "should" therefore be zero!

Why does this work when the Riemann integral did not?

1) We could cover the set $\{f(x)=1\}$ by countably many intervals. Whereas we use only finite coverings in the def. of the Riemann integral

2) We also measured the size of the set $\{f(x)=1\}$ first and then multiplied the "length" of the set by 1.
We don't care about oscillations of f!

Idea: Take $f(x)$ (arbitrary, not the f considered before) and split its domain into sets

We use this idea here!

$$S_k = \{x ; \ (k-1)\varepsilon < f(x) \leq k\varepsilon\}$$

and ~~…~~ let

$$s_\varepsilon(x) = \sum_{k=-N}^{N} \varepsilon k \ \chi_{S_k}(x)$$

where $N > \varepsilon \sup|f|$ and $\chi_{S_k}(x) = \begin{cases} 1 & x \in S_k \\ 0 & x \notin S_k \end{cases}$

Then $|f(x) - s_\varepsilon(x)| < \varepsilon$ and the ~~integral~~ area under f should be approximated, ~~by~~ within ε, by

$$I_\varepsilon = \sum_{k=-N}^{N} k\varepsilon \ m(S_k) \quad \text{where } m(S_k) = \text{"length of } S_k\text{"}$$

If we can find a way to measure S_k then I_ε is defined and $I_\varepsilon \to$ "area under f"

Okay, why don't we only change the def ~~off~~ the Riemann integral to allow countable partitions? Wouldn't that be enough? Then we could use $q_j \pm \frac{\varepsilon}{2^j}$ as the point of our partition and directly prove that $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ has integral zero! This does not work!

The reason is rather subtle. (paradoxical?)

But one might argue that

1) Start with $[0,1]$ one interval in the partition

2) subtract $(q_1 - \frac{\varepsilon}{2}, q_1 + \frac{\varepsilon}{2})$ this gives (at most) two intervals in the complement of $[0,1] \setminus (q_1 - \varepsilon/2, q_1 + \varepsilon/2)$

3) By induction assume that

$$[0,1] \setminus \bigcup_{j=1}^{k} \left(q_j - \frac{\varepsilon}{2^j}, q_j + \frac{\varepsilon}{2^j}\right) \quad \text{has at most}$$

$k+1$ intervals in its complement then subtracting

$$\left(q_{k+1} - \frac{\varepsilon}{2^{k+1}}, q_{k+1} + \frac{\varepsilon}{2^{k+1}}\right) \quad \text{will } \sout{\text{split}} \text{ add}$$

at most 1 new interval.

4) pass $k \to \infty$. then since ~~$k+1$~~ $\sup_{k \in \mathbb{N}} k+1 = \omega$ (cardinality of $\mathbb{N}$) it will follow that $[0,1] \setminus \bigcup_{j=1}^{\infty} (q_j - \frac{\varepsilon}{2^j}, q_j + \frac{\varepsilon}{2^j})$ has cardinality equal to ω. and is countable.

This *seems* like a proof.

But the same argument works if we consider $[0,1] \setminus \bigcup_{j=1}^{\infty} \{q_j\}$ = the irrationals = uncountable

So the argument cannot be right! We have no axiom that allows us to pass to the limit in step 4.

<u>Conclusion</u>: Allowing countable coverings does not fix the Riemann integral.

<u>Implication</u>: We need to develop a theory for measuring the "length" of sets.

We are now looking for a way to measure the "length" of a set $S \subset \mathbb{R}$.

That is, we need a function m: "subsets of $\mathbb{R}$" $\mapsto \mathbb{R}$.

We are not happy with just any function – it has to agree with what we mean by length

1) $m(a,b) = b-a = m([a,b])$ (length of an interval should be what we expect)

2) If $S \subset [a,b]$ then
$m(S) + m([a,b] \setminus S) = m([a,b]) = b-a$.

3) If S_j is a countable set of disjoint open sets then
$m\left(\bigcup_{j=1}^{\infty} S_j\right) = \sum_{j=1}^{\infty} m(S_j)$ (needed to "measure" $\mathbb{Q} \cap [0,1]$)

We will not get much further by discussing the properties of the domain of definition of m. We need to start figuring out what m should be. Fortunately we already have an idea:

Def: We define ~~that~~ m^*: "Sets of $\mathbb{R}$" $\mapsto \mathbb{R}_+$ to be

$$m^*(S) := \inf \sum_{j=1}^{\infty} (b_j - a_j)$$

where the infimum is taken over all open countable covers of S; that is ~~inter~~ over (a_j, b_j) s.t. $S \subset \bigcup_{j=1}^{\infty} (a_j, b_j)$. We say that m^* is the outer Lebesgue measure on $\mathbb{R}$.

Does this definition work?

Lemma: $m^*(a,b) = b-a = m^*([a,b])$

Proof: Let (a_j, b_j) be any cover of $[a,b]$ then, since $[a,b]$ is compact we may find a finite sub-cover (Heine-Borel thm), say $[a,b] \subset \bigcup_{j=1}^{N} (a_j, b_j)$. Clearly

$$\sum_{j=1}^{N} (b_j - a_j) \le \sum_{j=1}^{\infty} (b_j - a_j).$$

We may ~~start~~ therefore look at finite coverings of $[a,b]$.

Moreover we may assume that $(a_k, b_k) \not\subset \bigcup_{j \neq k} (a_j, b_j)$ (then throw out that interval)

So consider such a finite covering with N pieces, we will show that we can find a covering with smaller measure containing $N-1$ pieces.

To that end let (a_1, b_1) be an interval s.t. $a \in (a_1, b_1)$ (such exist since $\bigcup_{j=1}^{N} (a_j, b_j)$ covers $[a,b]$)

and let (a_2, b_2) contain b_1.

Then $(a_1, b_2), (a_3, b_3), (a_4, b_4), \dots (a_N, b_N)$ covers $[a,b]$ and

$$\sum_{j=1}^{N} b_j - a_j = \underbrace{b_1 - a_1 + b_2 - a_2}_{\substack{\geq b_2 - a_1 \\ \text{since } b_1 - a_2 \geq 0}} + \sum_{j=3}^{N} b_j - a_j \geq b_2 - a_1 + \sum_{j=3}^{\infty} b_j - a_j .$$

So we have a covering with one less interval and less measure.

It follows that the inf is achieved with one interval (a_1, b_1). ~~The~~ It easily follows that

$m^*([a,b]) = b-a$ (take $a_1 = a - \varepsilon$ $b_1 = b + \varepsilon$ and let $\varepsilon \to 0$).

For the open interval we notice that

if $m^*(a,b) = b - a - \delta$ then we could cover $[a,b]$ by the cover of (a,b) & $(a - \frac{\delta}{4}, a + \frac{\delta}{4})$ & $(b - \frac{\delta}{4}, b + \frac{\delta}{4})$

and get a contradiction to $m^*([a,b]) = b-a$.

Thus $m^*(a,b) \geq b-a \Rightarrow m^*(a,b) = b-a$. $\square$

Lemma: The outer measure is defined (maybe $=\infty$) for all subsets $S \subset \mathbb{R}$.

Proof: Since $\sum b_j - a_j \geq 0$ we get that

$$M = \left\{ \sum_{j=1}^{\infty} b_j - a_j ; \quad S \subset \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}$$ is a set of real numbers bdd from below thus either all the $\sum_{j=1}^{\infty} b_j - a_j$ diverges and $\inf M = \infty$ or $\exists c \in M \cap \mathbb{R}$ then $M \setminus \{\infty\}$ is a set of real numbers bdd from below and $\inf M$ exists. ∎

We now have a "good" function m^* defined on all sub-sets s.t. $m^*([a,b]) = m^*(a,b) = b-a$.
Can this mean that we are done? Answer: No!
We need to check that $m^*(\cup A_j) = \sum m^*(A_j)$ for disjoint A_j and $m(S) + m([a,b] \setminus S) = b-a$.
This is not true!

Proposition: There is no non-zero function $m \geq 0$ whatsoever from the sub-sets of $\mathbb{R}$ to $\mathbb{R}$ s.t.

1* m is translation invariant $m(S) = m(S+t)$ for all $t \in \mathbb{R}$ $(S+t = \{x+t;\ x \in S\})$

2* $m\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} m(A_j)$.

Proof: We will show that no measure exists on $[0,1)$, and consider all real numbers modulo 1.

Define the equivalence relation $x \sim y$ if $x-y \in \mathbb{Q}$. This divides $\mathbb{R}$ into equivalence classes. Using the axiom of choice we may define

$$V = \{\text{one } x \in [0,1) \text{ from each equivalence class}\}. \quad (V \text{ for Vitali})$$

And $V_j = V + q_j = \{x + q_j;\ q_j \in \mathbb{Q}\}$

where q_j is some denumeration of $\mathbb{Q}$.

Then $V_j \cap V_k = \phi$ (since ~~of~~ V only contains one element from each equivalence class $\Rightarrow x+q_j \neq y+q_k$ for $x,y \in V$)

Moreover $\bigcup_{j=1}^{\infty} V_j = [0,1)$ since $\bigcup_{j=1}^{\infty}\{x+q_j\}$ = eq class of x.

Therefore, if 2 holds,

$$m([0,1)) = m\left(\bigcup_{j=1}^{\infty} V_j\right) = \sum_{j=1}^{\infty} m(V_j) \qquad (1)$$

but by translation invariance $m(V_j) = m(V_k)$

Thus if $m(V_j) > 0$ then (1) will diverge

it follows that $m(V_j) = 0$ but then $m([0,1)) = 0$

contrary to m being non-zero. ∎

This is bad! - but not devastating.

We need to decrease the domain of definition of m^*. We still want some properties of the domain of definition - even though it cannot be the set of all subsets of $\mathbb{R}$.

Def: We say that $\mathcal{M}$, a set of subsets of S, is a σ-algebra iff

1) $S \in \mathcal{M}$, $\emptyset \in \mathcal{M}$
2) $A \in \mathcal{M} \Longrightarrow A^c = S \setminus A \in \mathcal{M}$
3) If $A_j \in \mathcal{M}$ and $A = \bigcup_{j=1}^{\infty} A_j$ then $A \in \mathcal{M}$ $(j \in \mathbb{N})$.

<u>Question:</u> What is the largest σ-algebra we can define m^* on so that

1) $m^*(S) + m^*([a,b] \setminus S) = b - a$
2) $m^*(\bigcup S_j) = \sum_{j=1}^{\infty} m^*(S_j)$?

Most important condition, we will use that to define the domain of m^*

Let us therefore use: $m^*(S) + m^*([a,b]\setminus S) = b-a$ as a definition... However, it makes more sense to define the domain of m^* to be all sets S s.t. $m^*(S\cap X) + m^*(S\setminus X) = m^*(X)$

Definition: We say that $S\subset \mathbb{R}$ is measurable if, for any $X\subset\mathbb{R}$,

$$m^*(S\cap X) + m^*(S\setminus X) = m^*(S). \qquad (1)$$

The measure that agrees with m^* on all measurable sets will be called the Lebesgue measure and denoted by m.

~~Thm: Every open set is measurable.~~

~~Proof.~~

Remark: $\cancel{m^*}(S\setminus X) = S^c \cap X$

Thus (1) can be written as

$$m^*(S\cap X) + m^*(S^c\cap X) = m^*(X)$$

Lemma: S measurable $\Rightarrow$ S^* measurable.

Now we have a measure m defined on the set of measurable sets. However, in order to use this measure for integration we need to show that m is defined on a σ-algebra. This requires some work.

We will begin to show that all open sets $S \subset \mathbb{R}$ are measurable. We begin to show that a single ~~...~~ open interval is open.

Lemma: Assume that U is open, $I = (a,b)$ then

$$m^*(U \cap (a,b)) + m^*(U^c \cap (a,b)) = m^*((a,b))$$

Proof: Since the union of any cover of $U \cap (a,b)$ and $U^c \cap (a,b)$ is a cover of (a,b) it follows that

$$m^*(U \cap (a,b)) + m^*(U^c \cap (a,b)) \geq m^*(a,b).$$

[The same is true for infinite sets as well $\sum m^*(A_j) \geq m^*(\bigcup A_j)$].

It is enough to show that

$$m^*(U \cap (a,b)) + m^*(U^c \cap (a,b)) \leq m^*(a,b).$$

To that end we argue in several steps

Step 1: We may write any open set $U \in \mathbb{R}$ as the countable union of open intervalls, $I_j = (a_j, b_j)$

Proof: Each component of U ~~...~~ ~~...~~ is an open interval that contain a $q \in \mathbb{Q}$.

Step 2: There exists, for every $\varepsilon > 0$, an N s.t.

$$\sum_{j=1}^{N} (b_j - a_j) + \varepsilon \geq m^*(U).$$

Proof Clear since $\sum_{j=1}^{\infty} b_j - a_j$ converges and $b_j - a_j \geq 0$

Step 3: U^c can be covered by finitely many closed intervals ~~$[a_j, b_j]$~~ $[b_j, a_k]$ (one $b_j = -\infty$, one $a_k = \infty$)

Proof Clearly $\bigcup_{j=1}^{N} (a_j, b_j) \subset U \Rightarrow$

$$U^c \subset \left(\bigcup_{j=1}^{N} (a_j, b_j) \right)^c$$ but the complement of finitely many open intervals is of the form described

Step 4: $$m^*(U \cap (a,b)) = \sum_{T_U} (b_j - a_j) + b - a - b^* + a^*$$

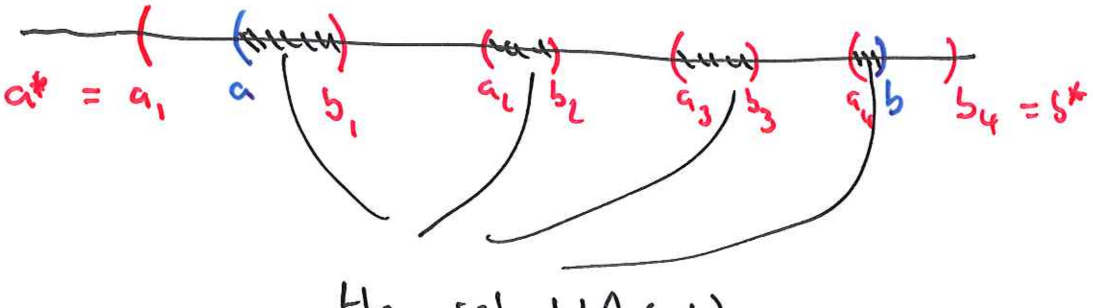
where $T = \{ j ; (a_j, b_j) \cap (a,b) \neq \emptyset \}$ and

$b^* =$ "the b_j, $j \in T$, s.t. $b_j \notin (a,b)$"

$a^* =$ " — — — — "

$a^* = a$ and $b^* = b$ if no such exsists.

Proof:



Clear from the picture.

Step 5: $$m^*(U^c \cap (a,b)) \leq \sum_{\substack{j \in T \\ j \leq N}} (a_{j+1} - b_j) + b - a - b^{**} + a^{**}$$

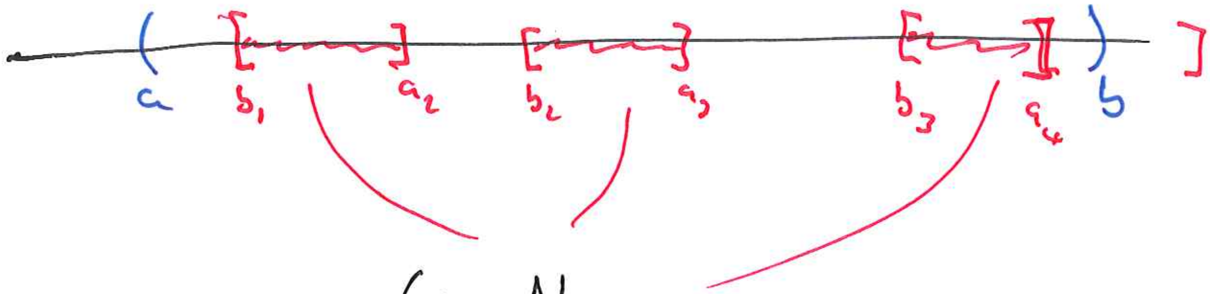
where $b^{**} =$ "the b_j s.t. $b \in [b_j, a_{j+1}]$" or b if no such

$a^{**} =$ " — a_{j+1} — $a \in [b_j, a_{j+1}]$" or a —"—

proof: $U^c \cap (a,b) \subset \left(\mathbb{R} \setminus \bigcup_{j=1}^{N} (a_j, b_j)\right) \cap (a,b)$

The set $\mathbb{R} \setminus \bigcup_{j=1}^{N} (a_j - b_j)$ consists of closed intervals $[b_j, a_{j+1}]$ that are disjoint.

This means that $m^*\left(\bigcup [b_j, a_{j+1}] \cap (a,b)\right) = \sum$ — —



$$\left(\mathbb{R} \setminus \bigcup_{j=1}^{N} (a_j, b_j)\right) \cap (a,b)$$

<u>Step 6</u>: $m^*(U \cap (a,b)) + m^*(U^c \cap (a,b)) =$

$$= \sum_{T_U} (b_j - a_j) + \sum_{\substack{T_U \\ j<N}} (a_{j+1} - b_j) + b - a \text{ — (not correct)}$$

$$\leq \underbrace{\left(\sum_{j>N} (b_j - a_j) \right)}_{< \varepsilon} + b - a \leq (b-a) + \varepsilon$$

for all $\varepsilon > 0$.

Lemma: Every open set U is measurable.

Proof: We need to show that if X is any set then $m^*(U \cap X) + m^*(U^c \cap X) = m^*(X)$.

We will assume that all sets have finite measure (or we could consider all sets intersected with $(-n, n)$ and then let $n \to \infty$)

Step 1 $\quad m^*(U \cap X) + m^*(U^c \cap X) \geq m^*(X)$.

This is clear since if

(a_j, b_j) covers $U \cap X$

(c_j, d_j) —"— $U^c \cap X$

then (a_j, b_j) and (c_j, d_j) covers X so

every cover (a_j, b_j), (c_j, d_j) that covers $U \cap X$ & $U^c \cap X$ covers X taking the inf among all covers of X will give smaller outer measure. (This is true irregardless of U is open or not!)

Step 2 $\quad m^*(U \cap X) + m^*(U^c \cap X) \leq m^*(X)$

To show this we pick an $\varepsilon > 0$ and an open cover $\underbrace{}_{(a_j, b_j) = I_j}$ for X s.t.

$$\sum_{j=1}^{\infty} b_j - a_j < m^*(X) + \varepsilon$$

Since $U \subset \mathbb{R}$ is open we may write

$$U = \bigcup_{j=1}^{\infty} \underbrace{(c_j, d_j)}_{=J_j}$$

{each rational $q \in U$ will be contained in a connected component and we may therefore biject intervalls of U with a subset of $\mathbb{Q} \Rightarrow U$ has countably many intervals}

Thus

$$m^*(U \cap X) \leq m^*\left(U \cap \left(\bigcup I_j\right)\right) \leq \left\{\begin{array}{l} m^* \text{ is} \\ \text{sub additive} \\ \text{as in step 1} \\ \text{works for countable} \\ \text{as well.} \end{array}\right\}$$

$$\leq \sum_j m^*(U \cap I_j)$$

and similarily $m^*(U^c \cap X) \leq \sum_j m^*(U \cap I_j)$

Thus

$$m^*(U \cap X) + m^*(U^c \cap X) \leq \sum_{j=1}^{\infty} \left(m^*(U \cap I_j) + m^*(U^c \cap I_j)\right) \quad (1)$$

Now, we would like to show that

$$m^*(U \cap I_j) + m^*(U^c \cap I_j) = m^*(I_j) = b_j - a_j \quad (\text{we want } *)$$

then (1) implies that $m^*(U \cap X) + m^*(U^c \cap X) \leq \sum_{j=1}^{\infty} b_j - a_j$

$$\leq m(X) + \varepsilon.$$

But (1) follows from the preceding Lemma.

(3)

Corollary: Every closed set is measurable.

proof: Since if K is closed, then $K^c = U$ is open thus, by previous thm,

$$m^*(X) = m^*(\underbrace{U}_{=K^c} \cap X) + m^*(\underbrace{U^c}_{=K} \cap X) = m^*(K^c \cap X) + m^*(K \cap X)$$

which is the definition of measurability.

Next we want to prove that if S_j are disjoint (countable collection) of measurable sets then $m^*(\bigcup_{j=1}^{\infty} S_j) = \sum_{j=1}^{\infty} m^*(S_j)$. This shows that $m = m^*$ defined on the measurable sets satisfies one of the properties that we claimed that any measure of interest should have.

The proof is done in the standard way in analysis. First we prove it for finite collections S_j, $1 \leq j \leq N$, and then the general case follows by passing to the limit.

Lemma: ~~Thm~~ (Countable additivity): Let S_j, $1 \leq j \leq N$, be a set of measurable sets. Then

$$m^*\left(\bigcup_{j=1}^{N} S_j\right) = \sum_{j=1}^{N} m^*(S_j).$$

<u>Proof:</u> By induction on N. If $N=2$ then we define $X = S_1 \cup S_2$ and by measurability of S_1 it follows that

$$m^*(X) = m^*(\underbrace{X \cap S_1}_{=S_1}) + m^*(\underbrace{X \setminus S_1}_{=S_2}) = m^*(S_1) + m^*(S_2)$$

Induction step; Assume $m^*\left(\bigcup_{j=1}^{n-1} S_j\right) = \sum_{j=1}^{n-1} m^*(S_j)$

then $m^*\left(\bigcup_{j=1}^{n} S_j\right) = \sum_{j=1}^{n} m^*(S_j)$.

proof of induction step: Denote $\hat{S} = \bigcup_{j=1}^{n} S_j$ then

by an argument as before, $X = \hat{S} \cup \underbrace{S_n}_{\text{measurable}}$

$$m^*(X) = m^*(\underbrace{X \cap S_n}_{=S_n}) + \underbrace{m^*(\underbrace{X \setminus S_n}_{=\hat{S}})}_{=\sum_{j=1}^{n-1} m^*(S_j)} = \sum_{j=1}^{n} m^*(S_j).$$

Thm (Countable additivity) Let S_j be a countable set of measurable sets. Then, is S_j disjoint

$$m^*\left(\bigcup_{j=1}^{\infty} S_j\right) = \sum_{j=1}^{\infty} m^*(S_j)$$

Proof: Since $m^*\left(\bigcup_{j=1}^{\infty} S_j\right) \geq m^*\left(\bigcup_{j=1}^{n} S_j\right) = \sum_{j=1}^{n} m^*(S_j)$

It follows that

$$m^*\left(\bigcup_{j=1}^{\infty} S_j\right) \geq \lim_{n\to\infty} \sum_{j=1}^{n} m^*(S_j) = \sum_{j=1}^{\infty} m^*(S_j).$$

But we also have

$$m^*\left(\bigcup_{j=1}^{\infty} S_j\right) \leq \sum_{j=1}^{\infty} m^*(S_j) \qquad \text{since}$$

any countable cover over all S_j is also etc.... .

$\blacksquare$

Next we need to show that the measurable sets forms a σ-algebra. Clearly, as we remarked before S measurable $\Rightarrow$ S^c measurable. Also ϕ and $\mathbb{R}$ are measurable. ~~Rome since~~

We need to show that if S_j are a countable collection of measurable sets then S_j is measurable.

Lemma: A finite set of measurable sets S_i

$\Rightarrow \bigcup_{i=1}^{N} S_i$ is measurable.

Proof: (Not very enlightening – skip during the lecture).

~~Let S_1~~ We will make an induction argument.

Let S_1 & S_2 be measurable, then

$$m^*(X) \underset{\substack{\uparrow \\ S_1 \text{ measurable}}}{=} m^*(X \cap S_1) + \underbrace{m^*(X \cap S_1^c)}_{\substack{\text{use this as} \\ X \text{ in } S_2 \text{ measurable}}} =$$

$$= \underbrace{m^*(X \cap S_1) + m^*((X \cap S_1^c) \cap S_2)}_{\geq m^*([X \cap S_1] \cup [(X \cap S_1^c) \cap S_2]) = \\ = m^*(X \cap [S_1 \cup S_2])} + m^*(\underbrace{(X \cap S_1^c) \cap S_2^c}_{= X \cap [S_1 \cup S_2]^c}) \geq$$

$$\geq m^*(X \cap [S_1 \cup S_2]) + m^*(X \cap [S_1 \cup S_2]^c).$$

The reverse inequality follows from sub-additivity $\quad m^*(A \cup B) \leq m^*(A) + m^*(B)$.

Thm: Let S_j be a countable ~~set~~ of collection of measurable sets. Then $\bigcup_{j=1}^{\infty} S_j = S$ is measurable.

Proof: We need to show that

$$m^*(\cancel{X}) \geq m^*(X \cap S) + m^*(X \cap S^c),$$

the other inequality follows from sub-additivity.

The trick is to notice that

$$S = \bigcup_{n=1}^{\infty} T_n$$

where $T_1 = S_1$ and $T_n = S_n \setminus T_{n-1}$

Then each T_n is measurable and the sets T_n are disjoint. Since T_n is measurable so is $L_n = \bigcup_{j=1}^{n} T_j$ and therefore

$$m^*(X) = m^*(X \cap L_n) + m^*(\underbrace{X \cap L_n^c}_{\supset X \cap S^c \text{ since } L_n \subset S}) \geq$$

$$\geq m^*(X \cap L_n) + m^*(X \cap S^c) =$$

$$= m^*\left(X \cap \left(\bigcup_{j=1}^{n} T_j\right)\right) + m^*(X \cap S^c) = \quad 1$$

To analyze $m^*(X \cap \bigcup_{j=1}^{n} T_j)$ we notice that ~~t~~ since T_j is measurable

$$m^*(X \cap \bigcup_{j=1}^{n} T_j) = m^*\left(\left[X \cap \bigcup_{j=1}^{n} T_j\right] \cap T_n\right) + m^*\left(\left[X \cap \bigcup_{j=1}^{n} T_j\right] \cap T_n^c\right)$$

$$= m^*(X \cap T_n) + m^*\left(X \cap \bigcup_{j=1}^{n-1} T_j\right) = \left\{\text{induction}\right\}$$

$$= \sum_{j=1}^{n} m^*(X \cap T_j).$$

We may therefore continue ① to get

$$m^*(X) \geq \sum_{j=1}^{n} m^*(X \cap T_j) + m^*(X \cap S^c).$$

Letting $n \to \infty$ gives

$$m^*(X) \geq \sum_{j=1}^{\infty} m^*(X \cap T_j) + m^*(X \cap S^c) \qquad ②$$

But $S = \bigcup_{j=1}^{\infty} T_j$ so by sub-additivity we may estimate $m^*(X \cap S) \leq \sum_{j=1}^{\infty} m^*(X \cap T_j)$ and thus ② may be estimated

$$m^*(X) \geq m^*(X \cap S) + m^*(X \cap S^c).$$

∎

$\underline{\text{Thm}}$ Assume that S is bdd and measurable. Then for every $\varepsilon > 0$ there exsists a finite union of intervals U s.t. $m(S \triangle U) < \varepsilon$

$\underline{\text{Proof}}$: Since S is measurable there exists an open cover $\hat{U}$ of S s.t.

$m(\hat{U}) \leq m(S) + \varepsilon/4$.

If $\hat{U} = \bigcup_{j=1}^{\infty} (a_j, b_j)$ then

there exist an N s.t. $\sum_{j=1}^{N} (b_j - a_j) + \frac{\varepsilon}{4} > m(\hat{U})$

Define $= \bigcup_{j=1}^{N} (a_j, b_j) =: U_\varepsilon$ then

$$m(S \triangle U) \leq \left\{\substack{\text{sub.}\\ \text{add}}\right\} \leq \underbrace{m(S \setminus U_\varepsilon)}_{< \underbrace{m(S\setminus \hat{U})}_{=0} + \underbrace{m(\hat{U} \setminus U_\varepsilon)}_{< \varepsilon/4}} + m(U_\varepsilon \setminus S) < \varepsilon/4 + m(U_\varepsilon \setminus S)$$

but $m(U_\varepsilon \setminus S) \leq m(\hat{U} \setminus S) < \varepsilon/4$

Thus $m(S \triangle U_\varepsilon) < \varepsilon/2 < \varepsilon$.

$U_\varepsilon =$ union of N intervalls. ∎

One final Theorem on measures

<u>Thm</u>: If S_j are measurable then

1) $S_j \subset S_{j+1}$ and $S = \bigcup_{j=1}^{\infty} S_j$ then

$$m(S) = \lim_{j \to \infty} m(S_j)$$

2) If $S_{j-1} \subset S_j$ and $S = \bigcap_{j=1}^{\infty} S_j$

then $$m(S) = \lim_{j \to \infty} m(S_j).$$

Proof. For 1) we define $T_j = S_{j+1} \setminus S_j$

then T_j are measurable and disjoint and

$S = \bigcup_{j=0}^{\infty} T_j$ (with $S_0 = \phi$) and $S_j = \bigcup_{i=1}^{j-1} T_j$

Thus, by countable additivity,

$$\lim_{j \to \infty} m(S_j) = \lim_{j \to \infty} m\left(\bigcup_{i=0}^{j-1} T_i\right) =$$

$$= \lim_{j \to \infty} \sum_{j=1}^{j-1} m(T_i) = \sum_{j=1}^{\infty} m(T_i) = m\left(\bigcup_{i=1}^{\infty} T_i\right) = m(S).$$

Proves 1)