(3.1) $X(n)$ ergodic consists of independent stochastic variables:

$E[X(n)] = m_x \quad E[X^2(n)] = \sigma_x^2 + m_x^2 \quad \forall n$

WSS + independence $\quad E[X(n_1)X(n_2)] = m_x^2 \quad \forall n_1, n_2 \quad n_1 \neq n_2$

$$r_x(k) = \begin{cases} \sigma_x^2 + m_x^2 & k=0 \\ m_x^2 & k \neq 0 \end{cases} \qquad Y(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(n-k)$$

Stationary!

(a) Variance σ_y^2 of $Y(n)$:

- $E[(Y-m_y)^2] = E[Y^2(n)] - m_y^2 \rightarrow \left\{ m_y = E[Y(n)] = E\left[\frac{1}{N}\sum_{k=0}^{N-1} X(n-k)\right] = \right.$

$= \{\text{linearity}\} = \frac{1}{N} E\left[\sum_{k=0}^{N-1} X(n-k)\right] = \{WSS\} = \frac{1}{N} N \cdot m_x = \underline{m_x} \Big\}$

$Y(n)$ is an unbiased estimator!

$$E[Y^2(n)] = E\left[\frac{1}{N}\sum_{k=0}^{N-1} X(n-k) \frac{1}{N}\sum_{j=0}^{N-1} X(n-j)\right] \underset{\text{linearity}}{=} \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} E[X(n-k)X(n-j)] =$$

$= \left\{\begin{array}{l}\text{to use independence we} \\ \text{need to separate the} \\ \text{cases with } k=j \text{ and} \\ k \neq j\end{array}\right\} = \frac{1}{N^2} \underbrace{\sum_{k=0}^{N-1}}_{N \text{ cases}} \cdot \underbrace{\sum_{j=k}}_{1 \text{ case}} E[X(n-k)^2] +$

$$+ \frac{1}{N^2} \underbrace{\sum_{k=0}^{N-1}}_{N \text{ cases}} \cdot \underbrace{\sum_{j \neq k}}_{(N-1) \text{ cases}} E[X(n-k)X(n-j)] = \frac{1}{N^2} \underbrace{\sum_{k=0}^{N-1}}_{N \text{ cases}} \underbrace{r_x(0)}_{\sigma_x^2 + m_x^2} + \frac{1}{N^2} \underbrace{\sum_{k=0}^{N-1}}_{N} \underbrace{\sum_{j \neq k}}_{N-1} \underbrace{r_x(k-j)}_{m_x^2} =$$

$$= \frac{1}{N^2} N(\sigma_x^2 + m_x^2) + \frac{1}{N^2} N(N-1) m_x^2 =$$

$$= \frac{1}{N}(\sigma_x^2 + m_x^2) + \frac{(N-1)m_x^2}{N} = \frac{1}{N}\sigma_x^2 + m_x^2$$

$$\sigma_y^2 = \frac{1}{N}\sigma_x^2 + m_x^2 - m_x^2 = \frac{1}{N}\sigma_x^2$$

Consistent estimator!

(i.e. $\sigma_y^2 \to 0$ when $N \to \infty$)

(b) σ_y^2 of $Y(n)$ by considering $Y(n)$ as an output of a linear system:

$$h(m) = \frac{1}{N}\sum_{k=0}^{N-1} \delta(m-k) \qquad r_y(\ell) = \underbrace{r_x(\ell) * h(\ell)} * h(-\ell)$$

$$\sigma_y^2 = r_y(0) - m_y^2 \qquad r_x(\ell) * h(\ell) = \frac{1}{N}\sum_{m=0}^{N-1} r_x(\ell-k)$$

$$h(-\ell) = \frac{1}{N}\sum_{k=0}^{N-1} \delta(-\ell-k) = \frac{1}{N}\sum_{k=0}^{N-1} \delta(\ell+k)$$

$$r_x(\ell) * h(\ell) * h(-\ell) = \frac{1}{N}\sum_{k=0}^{N-1} r_x(\ell-k) * \frac{1}{N}\sum_{j=0}^{N-1} \delta(\ell+j) =$$

$$= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} r_x(\ell+j-k) = r_y(\ell)$$

$$r_x(n) = \begin{cases} \sigma_x^2 + m_x^2 & n=0 \\ m_x^2 & n \neq 0 \end{cases}$$

since $\sigma_y^2 = r_y(0) - m_y^2 =$

$$r_y(0) = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} r_x(j-k) = \overbrace{\frac{1}{N^2}\sum_{k=0}^{N-1} r_x(0)}^{j=k} + \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{j\neq k} r_x(j-k) =$$

$$= \frac{1}{N^2} N \cdot r_x(0) + \frac{1}{N^2} N(N-1) m_x^2 = \frac{1}{N} r_x(0) + m_x^2 - \frac{1}{N} m_x^2$$

$= \frac{1}{N}\sigma_x^2 + m_x^2 \;/\; m_x^2 = m_y^2 \Rightarrow$ $\sigma_y^2 = \frac{1}{N}\sigma_x^2$

3.2

- Ergodic with ACF: $r_x(k) = \sigma_x^2 \delta(k) \Rightarrow$ uncorrelated with all others! + Ø mean

white!

σ_x^2 is estimated by the averaging $Z(n) = \frac{1}{N}\sum_{k=n-N+1}^{n} X^2(k)$

(a) Estimate $Z(n)$ of σ_x^2 unbiased?

$$E[Z(n)] = \frac{1}{N}\sum_{k=n-N+1}^{n} E[X^2(k)] = \frac{1}{N}\sum_{k=n-N+1}^{N} r_x(0) = \frac{N}{N} r_x(0) = \sigma_x^2$$

unbiased!

(b) σ_z^2 of $Z(n)$ as a function of the 4th order moment of $X(n)$:

$E[X^4(n)] = N_4$

$$E[Z^2(n)] = E\left[\frac{1}{N}\sum_{k=n-N+1}^{n} X^2(k) \frac{1}{N}\sum_{j=n-N+1}^{n} X^2(j)\right] = \{\text{linearity}\} =$$

$$= \frac{1}{N^2}\sum_{k=n-N+1}^{n}\sum_{j=n-N+1}^{n} E[X^2(k)X^2(j)] = \frac{1}{N^2}\sum_{k=n-N+1}^{n} \underbrace{E[X^4(k)]}_{j=k} +$$

$$+ \frac{1}{N^2}\sum_{k=n-N+1}^{N}\sum_{j\neq k} \underbrace{E[X^2(k)]E[X^2(j)]}_{\text{independence}} = \frac{\cancel{N}}{N^{\cancel{2}}} N_4 + \frac{\cancel{N}(N-1)}{N^{\cancel{2}}}\sigma_x^4$$

$$= \frac{N_4}{N} + \frac{N-1}{N}\sigma_x^4$$

$$\sigma_z^2 = E[Z(n)^2] - \sigma_x^4 = \frac{N_4}{N} + \frac{N-1}{N}\sigma_x^4 - \frac{N}{N}\sigma_x^4 =$$

$$= \frac{N_4}{N} - \frac{\sigma_x^4}{N} = \frac{1}{N}(N_4 - \sigma_x^4)$$

From charts we know that for a $\emptyset$ mean Gaussian random variable $E[X^4] = 3(E[X^2])^2$

(c) $\sigma_Z^2 = \frac{1}{N}(\mu_4 - \sigma_X^4) = \frac{1}{N}(3\sigma_X^4 - \sigma_X^4) = \frac{2}{N}\sigma_X^4$

$$\sigma_Z^2 \leq 0.01\,\sigma_X^4 \Rightarrow \frac{2}{N}\sigma_X^4 \leq 0.01\,\sigma_X^4$$

$$\frac{2}{0.01} \leq N \qquad \boxed{200 \leq N}$$

EXAM EXERCISE : PROBLEM 1

$U = X + Y$, $W = X - aY$ $\quad X: m_X, \sigma_X^2$; $Y: m_Y, \sigma_Y^2$ $\quad$ with covariance σ_{XY}

- Two variables are uncorrelated iff $C(U,W) = 0$
- Additionally: $C(U,W) = E[(U-m_U)(W-m_W)] = E[UW] - m_U m_W$

Hence, $C(U,W) = 0 \Rightarrow E[UW] = m_U m_W$

Note: It would be fulfilled if they were independent, but that can <u>NOT</u> be assumed!

$E[UW] = E[(X+Y)(X-aY)] =$

$= E[X^2 - aY^2 + YX - aXY] \xrightarrow{\text{linear}} E[X^2]$

$- aE[Y^2] + (1-a)E[XY] = \sigma_X^2 + m_X^2 - a(\sigma_Y^2 + m_Y^2) + (1-a)(\sigma_{XY} + m_X m_Y)$

$m_U m_W = (m_X + m_Y)(m_X - a m_Y)$

Then: $\sigma_X^2 + m_X^2 - a(\sigma_Y^2 + m_Y^2) + (1-a)(\sigma_{XY} + m_X m_Y) = (m_X + m_Y)(m_X - a m_Y)$

$\sigma_X^2 + \cancel{m_X^2} + \sigma_{XY} + \cancel{m_X} m_Y - a(\sigma_Y^2 + \cancel{m_Y^2} + \sigma_{XY} + \cancel{m_X} m_Y) = \cancel{m_X^2} - a\cancel{m_Y^2} + (\cancel{1}-\cancel{a})m_X m_Y$

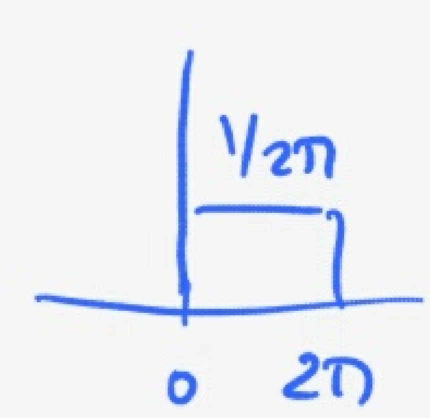
$$\sigma_X^2 + \sigma_{XY} = a(\sigma_Y^2 + \sigma_{XY}) \rightarrow a = \frac{\sigma_X^2 + \sigma_{XY}}{\sigma_Y^2 + \sigma_{XY}}$$

EXAM EXERCISE: PROBLEM 2

$X(t), Y(t)$ stochastic processes: $\quad X(t) = A_X\cos(\omega_0 t + \emptyset_X)$; $Y(t) = A_Y\cos(\omega_0 t + \emptyset_Y)$

$A_X, A_Y, \emptyset_X$ and $\emptyset_Y$ are independent.

$\emptyset_X$ and $\emptyset_Y$ are uniformly distributed $[0, 2\pi]$



$Z(t) \triangleq X(t)Y(t)$

$E[Z(t)] = E[X(t)Y(t)] = E[A_X A_Y \cos(\omega_0 t + \emptyset_X)\cos(\omega_0 t + \emptyset_Y)] \overset{\text{independence}}{=}$

$= E[A_X]E[A_Y]E[\cos(\omega_0 t + \emptyset_X)]E[\cos(\omega_0 t + \emptyset_Y)] =$

$= E[A_X]E[A_Y]\int_0^{2\pi}\cos(\omega_0 t + \emptyset_X)\frac{1}{2\pi}d\emptyset_X \int_0^{2\pi}\cos(\omega_0 t + \emptyset_Y)\frac{1}{2\pi}d\emptyset_Y =$

$= E[A_X]E[A_Y]\underbrace{\left(\frac{-\sin(\omega_0 t + \emptyset_X)}{2\pi}\right)\Big]_0^{2\pi}}_{\emptyset}\underbrace{\left(\frac{-\sin(\omega_0 t + \emptyset_Y)}{2\pi}\right)\Big]_0^{2\pi}}_{\emptyset} = 0$

$\cdot\ E[Z(t_1)Z(t_2)] = E[X(t_1)Y(t_1)X(t_2)Y(t_2)] = E[A_x A_Y \cos(\omega_0 t_1 + \phi_x) \cdot$
$\cos(\omega_0 t_1 + \phi_Y)\, A_x A_y \cos(\omega_0 t_2 + \phi_x)\cos(\omega_0 t_2 + \phi_Y)] = \{\text{independence}\}$
$= E[A_x^2]\,E[A_y^2]\ E[\cos(\omega_0 t_2 + \phi_x)\cos(\omega_0 t_1 + \phi_x)]\,E[\cos(\omega_0 t_2 + \phi_Y)$
$\cos(\omega_0 t_1 + \phi_Y)] = E[A_x^2]\,E[A_y^2]\,E\left[\frac{\cos(\omega_0(t_2 - t_1)) + \cos(\omega_0(t_2 + t_1) + 2\phi_x)}{2}\right]$
$\cdot E\left[\frac{\cos(\omega_0(t_2 - t_1)) + \cos(\omega_0(t_2 + t_1) + 2\phi_y)}{2}\right] =$

$$= E[A_x^2]\,E[A_y^2]\left(\frac{\cos(\omega_0(t_2 - t_1))}{2} + \int_0^{2\pi}\frac{\cos(\omega_0(t_2 + t_1) + 2\phi_x)}{4\pi}\,d\phi_x\right)\cdot$$

$$\cdot\left(\frac{\cos(\omega_0(t_2 - t_1))}{2} + \int_0^{2\pi}\frac{\cos(\omega_0(t_2 + t_1) + 2\phi_y)}{4\pi}\,d\phi_y\right) =$$

$$= E[A_x^2]\,E[A_y^2]\left(\frac{\cos(\omega_0(t_2 - t_1))}{2} + \cancelto{\phi}{\left[-\frac{\sin(\omega_0(t_2 + t_1) + 2\phi_x)}{8\pi}\right]_0^{2\pi}}\right)\left(\frac{\cos(\omega_0(t_2 - t_1))}{2}\right)$$

$$= E[A_x^2]\,E[A_y^2]\,\frac{\cos^2(\omega_0 \tau)}{4} \quad \text{with } \tau \triangleq t_2 - t_1$$

Hence $Z(t)$ is WSS