Handout #1

Katok - Hasselblatt

5.1.5 Proof of the Minimality Criterion

Now we prove that toral translations are minimal if and only if the translation vector is "completely irrational". This condition implies that γ_1 and γ_2 as well as their ratio are irrational. However, the condition is stronger than that, as the simple example of $\gamma_2 = 1 - \gamma_1$ with any irrational γ_1 shows.

The proof is considerably more elaborate than the simple argument from the proof of Proposition 4.1.1. The main idea, however, is the same: Unless the points on an orbit are aligned in a particular fashion, they will crowd all around, and this produces minimality. The main difference with the one-dimensional case is that then a "special alignment" simply meant finiteness of the orbit and hence periodicity, while now we have to capture an intermediate case and show that it appears only if orbits lie on parallel rational lines spiraling around the torus.

Proof of Proposition 5.1.2 We use additive notation. Such a translation is minimal if and only if the orbit of 0 is dense, because if $x \in \mathbb{T}^2$, then

$$T_\gamma(x) = x + \gamma = 0 + \gamma + x = T_\gamma(0) + x \pmod 1;$$

that is, the orbit $\mathcal{O}(x)$ of x is $T_x(\mathcal{O}(0))$, and therefore it is dense if and only if $\mathcal{O}(0)$ is because T_x is a homeomorphism. (This argument is the same as that in the proof of the more general Proposition 4.1.19.)

Pick $\epsilon > 0$ and consider the set D_ϵ of all iterates $T_\gamma^m(0)$ that are in the ϵ-ball $B(0, \epsilon)$ around 0. There are two possibilities:

(1) For some $\epsilon > 0$ the set D_ϵ is linearly dependent (that is, lies on a line).
(2) For any $\epsilon > 0$ the set D_ϵ contains two linearly independent vectors.

Below we prove three corresponding lemmas.

Lemma 5.1.8 *(2) ⇒ minimality.*

Lemma 5.1.9 *(1) ⇒ rational dependence.*

Lemma 5.1.10 *Rational dependence ⇒ (1).*

Minimality clearly excludes (1) and hence implies (2), so minimality is equivalent to (2). Thus minimality $\iff$ (2) $\iff$ *not* (1) $\iff$ rational independence. □

Proof of Lemma 5.1.8 This argument is similar to the proof of Proposition 4.1.1, albeit more complicated. It suffices to show that the orbit of 0 is dense. Take

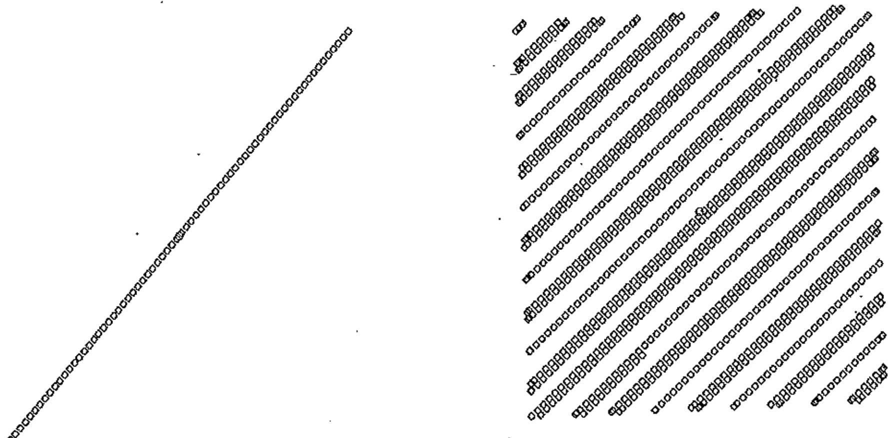

Figure 5.1.3. Dependent versus independent.

$\epsilon > 0$ and suppose $v_1, v_2 \in D_\epsilon$ are linearly independent. This means that they span a small parallelogram $\{av_1 + bv_2 \mid a, b \in [0, 1]\}$. The vertices of this parallelogram are all part of $\mathcal{O}(0)$: This is already known for 0, v_1, and v_2, and for $v_1 + v_2$ this is easy to see by representing v_1 and v_2 as $V_1 = 0 + m_1\gamma - k(m_1)$ and $V_2 = 0 + m_2\gamma - k(m_2)$ in $\mathbb{R}^2$, respectively, where $k(m_1)$ and $k(m_2)$ are those integer vectors for which $\|V_1\| < \epsilon$ and $\|V_2\| < \epsilon$. Then $V_1 + V_2 = 0 + (m_1 + m_2)\gamma - (k(m_1) + k(m_2)) = T_\gamma^{m_1+m_2}(0) \pmod 1$ and hence $v_1 + v_2 = T_\gamma^{m_1+m_2}(0)$.

Furthermore, the orbit of 0 contains all integer linear combinations of v_1 and v_2 [because $kV_1 + lV_2 = T_\gamma^{km_1+lm_2}(0) \pmod 1$]. Therefore, consider the tiling of the plane defined by the translates of $R := \{aV_1 + bV_2 \mid a, b \in [0, 1]\}$ by integer multiples of V_1 and V_2. This covers the plane with similar parallelograms, which have only boundary points in common, and every point of the plane is within ϵ of one of the vertices of these tiles (Figure 5.1.3). In particular, every point of $[0, 1] \times [0, 1]$ is within ϵ of some vertex, that is, every point of $\mathbb{T}^2$ is within ϵ of some point of $\mathcal{O}(0)$. According to the hypothesis of case (2), this is the case for any $\epsilon > 0$, that is, $\mathcal{O}(0)$ is dense in $\mathbb{T}^2$. □

Proof of Lemma 5.1.9 If 0 is periodic, then γ_1 and γ_2 are rational and we are done.

From now on assume that the orbit of 0 is infinite. Then for any $\epsilon > 0$ it contains two points $p = T_\gamma^m(0)$ and $q = T_\gamma^n(0)$ such that $\|q - p\| < \epsilon$. Then there are points $P = m\gamma \in \mathbb{R}^2$ and $Q = n\gamma + k \in \mathbb{R}^2$ such that $\epsilon > \|P - Q\| = \|m\gamma - n\gamma - k\| = \|(m - n)\gamma - k\|$, which means that $T_\gamma^{m-n}(0) - k \in B(0, \epsilon)$ and $D_\epsilon \neq \{0\}$ for all $\epsilon > 0$.

If $\epsilon > 0$ is as in (1), then $\{0\} \neq D_{\epsilon'} \subset D_\epsilon$ is linearly dependent for all $\epsilon' < \epsilon$. Thus D_ϵ lies on a unique line L through 0 given by an equation $ax + by = 0$.

Claim $\mathcal{O}(0)$ is dense on the projection of L. (See Figure 5.1.4.)

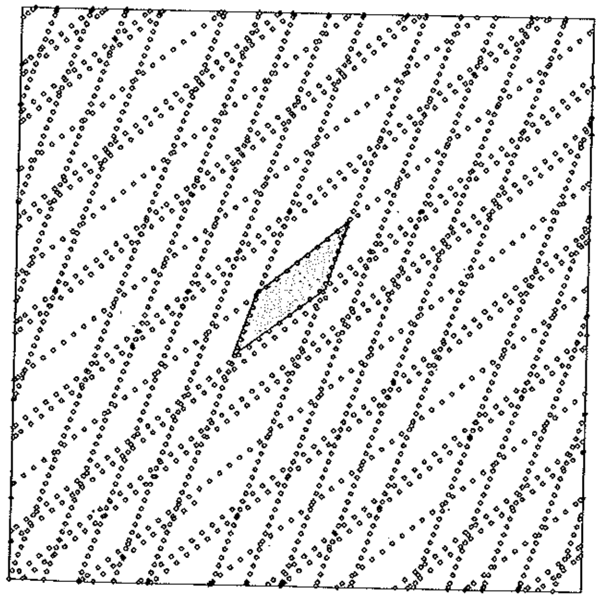

Figure 5.1.4. Density.

Since $D_{\epsilon'} \neq \{0\}$ for all $\epsilon' < \epsilon$, there are points $0 \neq p_{\epsilon'} \in D_{\epsilon'}$ and hence points $P = n\gamma - k \in L \cap B(0, \epsilon')$ (with $n \in \mathbb{Z}$, $k \in \mathbb{Z}^2$). But then $\{mP \mid m \in \mathbb{Z}\}$ is ϵ'-dense in L and projects into $\mathcal{O}(0)$.

Now a and b are rationally dependent because otherwise the slope of L is irrational, so the projection of L to $\mathbb{T}^2$ is dense and by the claim so is $\mathcal{O}(0)$. Therefore there exists $(k_1, k_2) \in \mathbb{Z}^2 \smallsetminus \{0\}$ such that $ak_1 - bk_2 = 0$. If $a = 0$ (or $b = 0$), then $ax + by = 0 \Leftrightarrow y = 0$ (or $x = 0$). Otherwise, multiply $ax + by = 0$ by $k_1/b = k_2/a$ to get $k_2 x + k_1 y = 0$, that is, we may take $a, b \in \mathbb{Z}$. If $n\gamma - k$ lies on the line $ax + by = 0$, then $an\gamma_1 - k_1 + bn\gamma_2 - k_2 = 0$ or $an\gamma_1 + bn\gamma_2 = k_1 + k_2$, which gives rational dependence. □

Proof of Lemma 5.1.10 Suppose $k_1\gamma_1 + k_2\gamma_2 = N \in \mathbb{Z}$ and divide by γ_1 to get $\gamma_2/\gamma_1 = (N - k_1)/k_2 =: s \in \mathbb{Q}$ (if $k_2 \neq 0$), that is, the iterates $(n\gamma_1, n\gamma_2)$ of 0 under repeated translation by γ lie on the line $y = sx$ with rational slope s. This projects to the torus as an orbit of the linear flow T_γ^t, which we found in Section 4.2.3 to be closed and hence not dense when $\gamma_2/\gamma_1 \in \mathbb{Q}$. Therefore the orbit of 0 under T_γ is not dense either, implying (1). (If $k_2 = 0$, then $k_1 \neq 0$ and the same argument works after exchanging x and y.) □

5.1.6 Uniform Distribution: The Kronecker–Weyl Method

The Kronecker–Weyl method of proving uniform distribution starting from trigonometric polynomials, then proceeding to continuous functions, and finally to characteristic functions, described in Section 4.1.6 also works in higher dimension. Again, to simplify notation we consider the two-dimensional case, leaving the extension to arbitrary dimension to the reader.

The *characters* corresponding to those in Section 4.1.6 are defined as group "homomorphisms" of $\mathbb{T}^2$ to S^1, where we view $\mathbb{T}^2$ as an additive group (as described at the beginning of this chapter) and S^1 is considered as the group of complex numbers of absolute value one with multiplication as the group operation. A homomorphism is a map that preserves this group structure, that is, the image of the sum of two elements is the product of their images. To be specific, if we use additive notation for the torus, then the characters have the following form:

$$c_{m_1,m_2}(x_1, x_2) = e^{2\pi i(m_1x_1+m_2x_2)} = \cos 2\pi(m_1x_1 + m_2x_2) + i \sin 2\pi(m_1x_1 + m_2x_2),$$

where (m_1, m_2) is any pair of integers. Finite linear combinations of characters are called *trigonometric polynomials* beacause they also can be expressed as finite linear combinations of sines and cosines. Characters are *eigenfunctions* for the translation because

$$c_{m_1,m_2}(T_\gamma(x_1, x_2)) = e^{2\pi i(m_1(x_1+\gamma_1)+m_2(x_2+\gamma_2))} = e^{2\pi i(m_1\gamma_1+m_2\gamma_2)} c_{m_1,m_2}(x_1, x_2).$$

A crucial observation for our purposes is that, since γ_1, γ_2, and 1 are rationally independent, that is, $m_1\gamma_1 + m_2\gamma_2$ is never an integer unless $m_1 = m_2 = 0$, the *eigenvalue* $e^{2\pi i(m_1\gamma_1+m_2\gamma_2)} \neq 1$ unless $m_1 = m_2 = 0$.