Handout # 4 Ulcigrai - Marklof

A.5 Extra Section: Quadratic maps: attracting and repelling fixed points

Let us motivate the study of the quadratic maps from a simple (but very rich!) model of population growth in biology. Let x be the size of a population. Consider a *discrete* model, in which the population size grows at discrete time intervals, say, for example, every year[2]. If the population y during the following year is depends only on the present population x, we can model the growth by an equation of the form $y = f(x)$. Thus, $f^n(x)$ gives the population after n time intervals. The simplest model is

$$f(x) = \mu x$$

where $\mu > 0$ is a parameter which gives the fertility of the species. By induction, one sees that

$$f^n(x) = \mu^n x,$$

and since

$$\lim_{n\to\infty} \mu^n x = 0 \quad \text{if } \mu < 1, \qquad \lim_{n\to\infty} \mu^n x = +\infty \quad \text{if } \mu > 1,$$

the population either becomes extinct (if $\mu < 1$, there are not enough children), or it grows exponentially (if $\mu > 1$).

[2]This is the case for certain populations, for example of butterflies, which are seasonal and the size of the population the next summer depends only on the previous generation the summer before.

This model is too simple because in reality resources are limited and if the environment is overcrowded, there is not enough food to support exponential growth. One can assume that there is an upper value L for the size of the population which can be supported by the environment. The second simplest model often used by biologist is:

$$f(x) = \mu x(L - x).$$

If x is very close to L, the population grows very slowly. If x/L is small though, we still have exponential growth until we approach the value L. If $x = L$, $f(x) = 0$: there is not enough food and all population die before reproducing. If does not make sense to consider values of x bigger than L (then $f(x)$ is negative).

For convenience, we can rescale variables and assume that $L = 1$. One should think of $x \in [0,1]$ as a *percentage*, giving the population size as a percentage of the maximum value L. We obtain the following map, that we call f_μ since it depends on the parameter μ.

$$f_\mu(x) = \mu x(1 - x).$$

As μ changes, we get a family of maps known as the *quadratic family*. It is also called *logistic family*[3]. Values of the parameter which are studied are $0 < \mu \leq 4$. For these values,

$$f_\mu([0,1]) \subset [0,1],$$

so we can iterate the map. If $\mu > 4$, some of the points in $[0,1]$ are mapped *outside* of the domain $[0,1]$, so we cannot iterate our function (one can nevertheless restrict the domain and consider f only on the set of points whose forward iterates all belong to $[0,1]$, which turns out to be a fractal, see Extra A.5.1). Let us here consider here only the parameters $0 \leq \mu \leq 4$ and let us investigate orbits of f_μ when $0 \leq \mu \leq 4$. Remember that the orbit $\mathcal{O}_f^+(x)$ can be thought of describing the behavior of an initial population x under this model. For example we would like to know if the values $f^n(x)$ after large time n stabilize, and in this case to which values, or if it oscillates.

Consider as an example the map corresponding to $\mu = 5/2$:

$$f(x) = \frac{5}{2}x(1 - x).$$

The graph of f can be drawn noticing that it is a parabola, $f(0) = f(1) = 0$ and that the derivative $f'(x) = 5/2 - 5x$ is zero at $x = 1/2$ for which $f(1/2) = 5/8$. In particular, f maps $[0,1]$ to $[0,1]$. See Figure A.2.

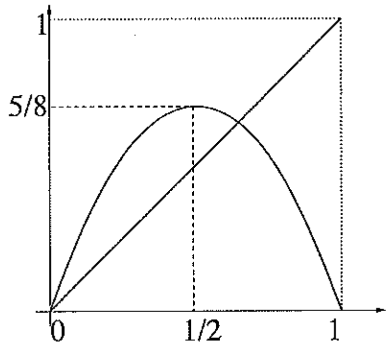

Figure A.2: The graph of the quadratic map $f(x) = \frac{5}{2}x(1-x)$.

Fixed points of f are solutions of the equation $f(x) = x$. In this case, solutions of $5/2x - 5/2x^2 = x$ or equivalently $x(3/2 - 5/2x) = 0$ are only $x = 0$ and $x = 3/5$.

Remark A.5.1. *Graphically, fixed points are given by considering the intersections of the graph of f, that is the set $\mathscr{G} = \{(x, f(x)), x \in X\}$ with the diagonal $\Delta = \{(x,x), \ x \in X\}$ and taking their horizontal components. Equivalently, x is a fixed point if and only if $(x,x) \in \mathscr{G}$.*

To have an idea of the empirical behavior of an orbit $\mathcal{O}_f(x)$ one can use the following graphical method.

[3]Logistic comes from the French *logistique*, which is derived from the *lodgement* of soldiers. The equations were introduced by the sociologist and mathematician Verhulst in 1845.

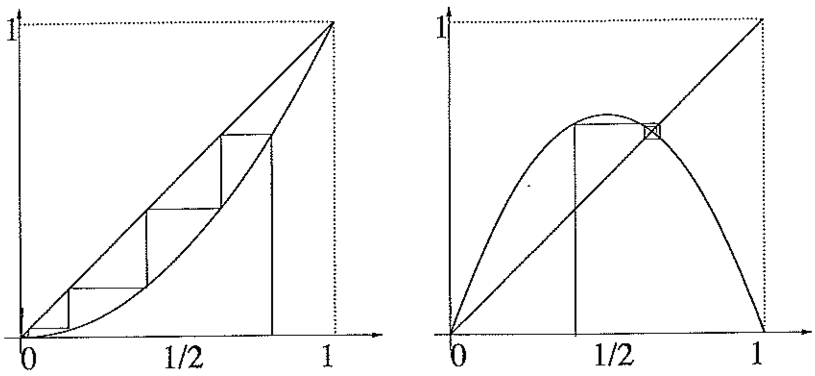

Figure A.3: Examples of graphical analysis.

Graphical Analyses

- Draw the graph $\mathscr{G}$ of f and the diagonal $\Delta = \{(x,x), \quad x \in X\}$;
- Start from $(x,0)$. Move vertically up until you intersect the graph $\mathscr{G}$ of f at $(x, f(x))$;
- Move horizontally until you hit the diagonal Δ, at $(f(x), f(x))$; the horizontal projection is now $f(x)$;
- Move vertically to hit the graph $\mathscr{G}$, and then again horizontally to hit the diagonal;
- Repeat the step above.

At step $n \geq 1$ one hits the graph at $(f^{n-1}(x), f^n(x))$ and the diagonal at $(f^n(x), f^n(x))$. Thus the horizontal projections of the points obtained give the orbit $\mathcal{O}_f(x)$. This method allows to guess what is the asymptotic behavior of $\mathcal{O}_f(x)$. You can for example look whether the points $(f^n(x), f^n(x)) \in \Delta$ which you obtain are converging towards (or diverging from) a fixed point $(x,x) \in \Delta$. In our example, the graphical analysis shows that values of $f^n(x)$ tend to oscillate around 3/5 but tend to stabilize toward it, see the right picture in Figure A.3.

Exercise A.5.1. *Use the graphical analysis to find fixed points and study the behavior or orbits nearby for the following functions:*

(1) $g(x) = x - x^2$ *for* $0 \leq x \leq 1$;

(2) $g(x) = 2x - x^2$ *for* $0 \leq x \leq 1$;

(3) $g(x) = -x^3$ *for* $-\infty \leq x \leq \infty$;

Exercise A.5.2. *Draw the behavior of the orbits of* $f(x) = 5/2x(1-x)$ *near* 0 *and near* $3/5$.

The graphical analysis of f_μ for $\mu = 5/2$ suggested that for any $x \in (0,1)$ the values of $f^n(x)$ tend to oscillate around 3/5 but to stabilize toward it. Let us prove that this is indeed the case.

Given a ball $U := B(x,\epsilon) = \{y | \, d(x,y) < \epsilon\}$, let $\overline{U} := \overline{B(x,\epsilon)}$ be the closed ball $\{y | \, d(x,y) \leq \epsilon\}$. Note that in $X = [0,1]$ the ball $B(x,\epsilon)$ is simply a open interval $(x-\epsilon, x+\epsilon)$ and the closed ball is the corresponding closed interval $[x-\epsilon, x+\epsilon]$. We give here the definition using balls, since this definition holds more in general in any *metric space*, that is a space where there is a notion of distance (see Chapter 2).

Definition A.5.1. *We say that a fixed point x is an* attracting *fixed point if there exists a ball* $U := B(x,\epsilon)$ *around x such that*

$$f(\overline{U}) \subset U, \quad \textit{and} \quad \cap_{n\in\mathbb{N}} f^n(U) = \{x\}.$$

We say that a fixed point x is an repelling *fixed point if there exists a ball* $U := B(x,\epsilon)$ *around x such that*

$$\overline{U} \subset f(U), \quad \textit{and} \quad \cap_{n\in\mathbb{N}} f^{-n}(U) = \{x\}.$$

[Note that here f is not necessarily invertible. By $f^{-1}(U)$ we mean the set-preimage of the set U: $f^{-1}(U)$ is the set of all points y such that $f(y) \in U$ (that is, all preimages of U).]

Exercise A.5.3. *Show that if f is invertible, x is an attracting fixed point if and only if it is a repelling fixed point for f^{-1} and viceversa.*

When X is an interval in $\mathbb{R}$ there is an easy criterion to determine whether a fixed point is attracting or repelling.

Theorem A.5.1. *Let $X \subset \mathbb{R}$ be an interval and let $f : X \to X$ be a differentiable function with continuous derivative. Let $x = f(x)$ be a fixed point.*

(1) If $|f'(x)| < 1$, then x is an attracting fixed point. More precisely, we will find and open ball U such that $f(\overline{U}) \subset U$ and for all $y \in U$ we have

$$\lim_{n\to\infty} f^n(y) = x.$$

(2) If $|f'(x)| > 1$, then x is a repelling fixed point.

Remark A.5.2. *Note that if $|f'(x)| = 1$ it is not possible to determine just from this information whether the fixed point is repelling or attracting.*

Proof. Let us prove (1). Since f' is continuous and $|f'(x)| < 1$, there exist an $\epsilon > 0$ such that for all $y \in [x-\epsilon, x+\epsilon] = \overline{B(x,\epsilon)}$ we have $|f'(y)| < \rho < 1$. Then for all $y \in \overline{B(x,\epsilon)}$, since $f(x) = x$, by Mean Value Theorem there exists $\xi \in B(x,\epsilon)$ such that

$$|f(y) - x| = |f(y) - f(x)| = |f'(\xi)||y - x| \le \rho|y - x| \le \rho\epsilon.$$

This givens that $f(y) \in (x-\epsilon, x+\epsilon)$ for all $y \in \overline{B(x,\epsilon)}$. Thus $f(\overline{B(x,\epsilon)}) \subset B(x,\epsilon)$.

Let us prove by induction that

$$|f^n(y) - x| \le \rho^n\epsilon.$$

We already proved it for $n = 1$. Assume that it holds for $n \ge 1$. Then, applying mean value as before, for all $y \in B(x,\epsilon)$, there exists $\xi \in B(x,\epsilon)$ such that

$$|f^{n+1}(y) - f^{n+1}(x)| = |f(f^n(y)) - f(f^n(x))| = |f'(\xi)||f^n(y) - f^n(x)| \le \rho|f^n(y) - x|$$

and by the induction assumption, since $f^{n+1}(x) = x$, this gives

$$|f^{n+1}(y) - x| = |f^{n+1}(y) - f^{n+1}(x)| \le \rho|f^n(y) - x| \le \rho^{n+1}\epsilon.$$

Since $\rho < 1$, $\lim_{n\to\infty} \rho^n = 0$. Thus, we get at the same time that

$$\lim_{n\to\infty} f^n(y) = x \quad \text{and} \quad \cap_{n\in\mathbb{N}} f^n(B(x,\epsilon)) = \{x\}.$$

The proof of part (2) is similar. □

Exercise A.5.4. *Prove part (2) of Theorem A.5.1.*

Exercise A.5.5. *In our example $f(x) = 5/2x(1-x)$, one can check that $f'(0) = 5/2$ and $f'(3/5) = -1/2$ so that 0 is a repelling fixed point and 3/5 is an attracting fixed point. Moreover, for each $\delta > 0$, $f([\delta, 1]) \subset (0,1)$ and all points points converge to 3/5.*

The dynamics of this quadratic map is then very simple, it is an *attracting-repelling dynamics.* If one changes 5/2 with 4 and considers the map $f(x) = 4x(1-x)$, the behavior is completely different and much more chaotic.

Exercise A.5.6. *Program a computer program to plot some iterates of $f(x) = 4x(1-x)$ at some points. Is there any pattern? Compare with the case $f(x) = \mu x(1-x)$ with $0 < \mu < 3$.*

For $0 \le \mu < 3$, the behavior of the maps

$$f_\mu(x) = \mu x(1-x).$$

the quadratic family is also very simple and similar to the one for $\mu = 5/2$. There are only attracting and repelling fixed points and all the other points are attracted or repelled.

Exercise A.5.7. *Consider the quadratic family f_μ for $\mu \in [0,4]$.*

(1) Check that for $\mu \in [0,4]$ the interval $I = [0,1]$ is mapped to itself, i.e. $f_\mu(I) \subset I$.

(2) Check that the fixed points of f_μ are 0 and $1 - 1/\mu$.

(3) Determine for which values of μ each of them is a repelling or attracting fixed point according to the criterion in Theorem A.5.1.

What happens for $3 < \mu \leq 4$? When μ is slightly bigger than 3, one finds that instead than an attracting fixed point, there is an attracting orbit: the values oscillate and tend to converge to a periodic cycle. The periods of the periodic orbit happen to *double* as one moves the parameter, showing a phenomenon known as *period doubling* (see the Extra A.5.2 if you want to experiment it). The dynamics as μ approaches 4 is very rich and displays interesting new chaotic phenomena sometimes called *the route to chaos*. Finally, the dynamics of f_μ for $\mu = 4$ is very chaotic and turns out to be very similar to the dynamics of the doubling map that we will see in the next lecture[4], see §1.4.

A.5.1 Quadratic maps for $\mu > 4$, Cantor fractals and invariant sets

For $\mu > 4$ the interval I is no longer invariant under f_μ, i.e. there are points which are mapped outside I. It is still possible to consider the dynamics of f_μ, but one has to restrict the domain to an invariant subset of $[0,1]$, i.e. to the set C of the form

$$C = \cap_{n\in\mathbb{N}} f_\mu^{-n}(I). \tag{A.2}$$

If $x \in C \subset I$, for each $n \in \mathbb{N}$, $f_\mu^n(x) \in I$, so that $\mathcal{O}_{f_\mu}(x) \subset I$. This set is called the *invariant set* of the map f_μ and it turns out to be non-empty. The map f_μ on the space $X = C$ gives a well defined dynamical system, since $f(C) \subset C$ and for any $x \in C$ we can iterate f forever. The set C, though, has a quite complicated fractal structure: it is a *Cantor set.*

The best known Cantor set is the *middle third Cantor set*, that can be defined by the following iterative construction. Start at level zero from $C^0 = [0,1]$. Divide the interval into 3 equal thirds and remove the open middle interval, that is $(1/2, 2/3)$. The remaining set, that we call C^1, consists of the 2 closed intervals $[0,1/3]$ and $[2/3,1]$, each of length $1/3$. To go to the next stage, divide each of these two intervals into 3 equal intervals and remove the two middle thirds. You are left with is the set

$$C^1 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

which consists of 4 intervals of size $1/9$. Iterating this construction, at step n we get a set C^n which consists of 2^n intervals of size $1/3^n$ (see Figure A.4). The intersection $C_3 = \cap_n C^n$ of all these sets is not empty and

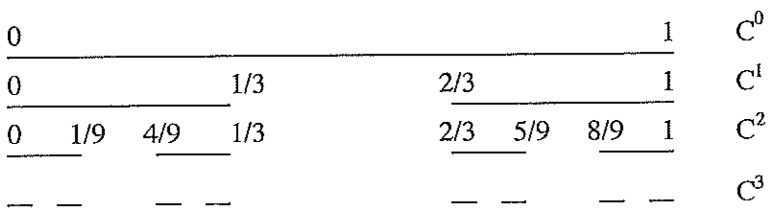

Figure A.4: The construction of the middle-third Cantor set.

is the middle-third Cantor set. The Cantor set C_3 is self-similar in the following sense: if you consider for example $C_3 \cap [0,1/3]$, and blow it up by applying the map $x \mapsto 3x$, you get back the same Cantor set. This self-similarity happens at all scales and is responsible for the fractal nature of C_3.

Let $f = f_\mu$ be a quadratic map with $\mu > 4$ and let us describe its invariant set iteratively. One can see that the points x such that $f(x) \in I$ belong to the two disjoint subintervals, say I_1 and I_2 such that

$$f^{-1}([0,1]) = I_1 \cup I_2.$$

[4] It is possible to show that these two maps are *conjugated* in the sense defined in §1.4 and hence they have similar dynamical properties, for example the same number of periodic points.

The points for which $f(x) \in I$ and $f^2(x) \in I$ belong to

$$f^{-1}(I_1) \cup f^{-1}(I_2)$$

which consists of 4 disjoint intervals, two obtained by removing a central subinterval from I_1 and the other two obtained by removing a central interval from I_2. Continuing like this, one can see that the points which can be iterated n times belong to a disjoint union of 2^n intervals. The set of points which can be iterated infinitely many times can be obtained by iterating this construction. What is left by intersecting all the disjoint unions of 2^n intervals is also a *Cantor set* and has a fractal structure.

A.5.2 Quadratic maps for $3 < \mu < 4$, experimenting period doubling

The dynamics of the quadratic family for $\mu \in [3, 4)$ is very rich and displays interesting chaotic phenomena, known as *period doubling* or sometimes called *the route to chaos.* We will not treat them in this course, but if you can write a simple computer program, try the following exercises to get a sense of it:

Exercise A.5.8. *Starting with $x = 0.001$, iterate f_μ for $\mu = 2.9$ and $\mu = 3$ until you discern a clear pattern.*

The population in both cases settles down, but for $\mu = 3$ there are fairly substantial oscillations of too large and too small population which die out slowly.

Exercise A.5.9. *Starting with $x = 0.66$, iterate f_μ for $\mu = 3.1$ until you discern a clear pattern.*

Here oscillations do not die out. It is possible to prove that they are stable, whatever the starting data, the population keeps running into overpopulation every other year.

Exercise A.5.10. *Starting with $x = 0.66$, iterate f_μ for $\mu = 3.45$ and $\mu = 3.5$ until you discern a clear pattern.*

Now oscillations involve four population sizes: big, small, big, small in a 4-cycle.

Exercise A.5.11. *Try to explore the behavior of f_μ increasing slowly the parameter μ for values slightly larger than $\mu = 3.5$. Try to see if you can find oscillations of the population size of size 8, 16 and other multiples of 2.*

This phenomenon is known as *period doubling.*

Exercise A.5.12. *Starting with $x = 0.5$, iterate f_μ for $\mu = 3.83$ until you discern a clear pattern.*

You will find that here there are oscillations, but not more of a period multiple of 2, but of period 3... then the periods of the oscillations will become $2 \cdot 3$, $2^2 \cdot 3$, $2^3 \cdot 3 \ldots$..., that is there will be a new period doubling cascade. The behavior of this family has been object of fascinating research and what is sometimes called *route to chaos* is know well understood.

If you want to explore more about the behavior of the quadratic family for $3 \leq \mu \leq 4$ we suggest the following references:

[4] R. Devaney *Chaotical Dyanamical Systems*, Springer

[5] K. Alligood, T. Sauer, j. Jorke, *Chaos: an Introduction to Dyanamical Systems*, Springer

