Handout #5 from Ulcigarai - Marklof

1.4 Baker's map

Let $[0,1)^2 = [0,1) \times [0,1)$ be the unit square. Consider the following two dimensional map $F : [0,1)^2 \to [0,1)^2$

$$F(x,y) = \begin{cases} (2x, \frac{y}{2}) & \text{if } 0 \le x < \frac{1}{2}, \\ (2x-1, \frac{y+1}{2}) & \text{if } \frac{1}{2} \le x < 1. \end{cases}$$

Geometrically, F is obtained by cutting $[0,1)^2$ into two vertical rectangles $R_0 = [0,1/2) \times [0,1)$ and $R_1 = [1/2,1) \times [0,1)$, stretching and compressing each to obtain an interval of horizontal width 1 and vertical height 1/2 and then putting them on top of each other. The name *baker's map* comes becouse this mimic the

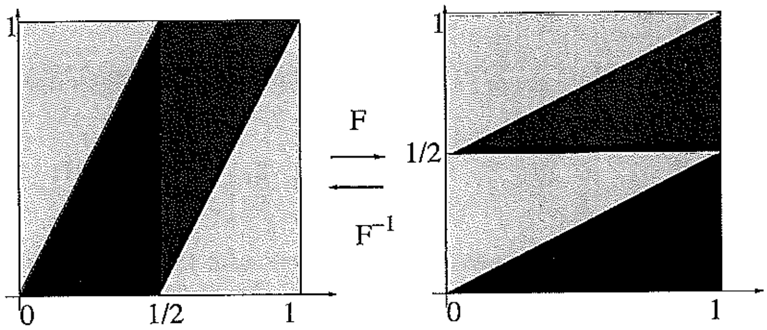

Figure 1.4: The action of the baker's map.

movement made by a baker to prepare the bread dough[9]. Similar maps are often use in industrial processes since, as we will see formally later, they are very effective in quickly *mixing*.

Remark 1.4.1. *Notice while the horizontal direction is stretched by by a factor* 2, *the vertical direction is contracted by a factor* 1/2.

The baker map is invertible. The inverse of the map F can be explicitely given by

$$F^{-1}(x,y) = \begin{cases} (\frac{x}{2}, 2y) & \text{if } 0 \le y < \frac{1}{2}, \\ (\frac{x+1}{2}, 2y-1) & \text{if } \frac{1}{2} \le y < 1. \end{cases}$$

Geometrically, F^{-1} cuts X into two horizontal squares and stretches each of them to double the height and divide by two the width and then places them one next to each other (in Figure 1.4, the right square now gives the departing rectangle decomposition and the left square shows the images of each rectangle under F^{-1}).

Unlike in the case of the doubling map, we now have to be more careful in identifying F as a map on $X = (\mathbb{R}/\mathbb{Z})^2$. The map $F : X \to X$

$$F(x,y) = \begin{cases} \left(\{2x\}, \frac{\{y\}}{2}\right) & \text{if } 0 \le \{x\} < \frac{1}{2}, \\ \left(\{2x\}, \frac{\{y\}+1}{2}\right) & \text{if } \frac{1}{2} \le \{x\} < 1. \end{cases}$$

is well defined; here $\{y\} = y \mod 1$ denotes the fractional part of y.

If you compare the definition of the baker map F with the doubling map f in the previous section, you will notice that the horizontal coordinate is trasformed exactly as f. More precisely one can show that F is an extenstion of f (i.e. there is a semiconjugacy ψ such that $\psi \circ F = f \circ \psi$, see Exercise below). Extensions of non-invertible maps which are invertible are called *intervible extensions* or *natural extensions.*

Exercise 1.4.1. *Show that the doubling map* $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ *and the baker map* $F : (\mathbb{R}/\mathbb{Z})^2 \to (\mathbb{R}/\mathbb{Z})^2$ *are semi-conjugated and the semi-conjugacy is given by the projection* $\pi : (\mathbb{R}/\mathbb{Z})^2 \to \mathbb{R}/\mathbb{Z}$ *given by* $\pi(x,y) = x$.

[9] There are other versions of the baker's map where the dough is not cut, but folded over.

To study the doubling map, we introduced the one-sided shift on two symbols ($\sigma^+ : \Sigma^+ \to \Sigma^+$). To study the baker map is natural to introduce the *bi-sided shift* on two symbols, that we now define.

Let $\Sigma = \{0,1\}^{\mathbb{Z}}$ be the set of all bi-infinite sequences of 0 and 1:

$$\Sigma = \{(a_i)_{i=-\infty}^{\infty}, \quad a_i \in \{0,1\}\}.$$

A *point* $\underline{a} \in \Sigma^+$ is a bi-sided sequence of digits 0,1, for example

$$\ldots 0,0,1,1,0,1,0,0,0,1,1,0,1,1,1,1,0,0,1,0,1,0\ldots$$

The (bi-sided) *shift* map σ is a map $\sigma : \Sigma \to \Sigma$ which maps a sequence to the *shifted* sequence:

$$\sigma((a_i)_{i=\infty}^{\infty}) = (b_i)_{i=-\infty}^{\infty}, \qquad \text{where} \quad b_i = a_{i+1}. \tag{1.8}$$

The sequence $(b_i)_{i=-\infty}^{\infty}$ is obtained from the sequence $(a_i)_{i=-\infty}^{\infty}$ by shifting all the digits one place to the left. For example, if

$$\begin{array}{rcccccccccccccccccc} (a_i)_{i=-\infty}^{\infty} & = & \ldots & 0, & 0, & 1, & 1, & 0, & 1, & 0, & 0, & 0, & 1, & 1, & 0, & 1, & 1, & 1, & 1, & \ldots \\ (b_i)_{i=-\infty}^{\infty} & = & 0, & 0, & 1, & 1, & 0, & 1, & 0, & 0, & 0, & 1, & 1, & 0, & 1, & 1, & 1, & 1, & \ldots & \end{array}$$

Note that while the one-sided shift σ^+ was not invertible, because we were throwing away the first digit a_1 of the one-sided sequence $(a_i)_{i=1}^{\infty}$ before shifting to the left, the map σ is now invertible. The inverse σ^{-1} is simply the shift to the right.

One can show that the baker map F and the bi-sided shift σ are semi-conjugate if σ is restricted to a certain shift-invariant subspace (see Theorem 1.4.1 below).

In the case of the doubling map, the key was to use binary expansion. What to use now? We can get a hint of what is the semi-conjugacy using itineraries and trying to understand sets which share a common part of their itinerary.

Let R_0 and R_1 be the two basic rectangles

$$R_0 = \left[0, \frac{1}{2}\right) \times [0,1), \qquad R_1 = \left[\frac{1}{2}, 1\right) \times [0,1).$$

(See Figure 1.4, left square: R_0 is the left rectangle, R_1 the right one.)

The *(bi-infinite) itinerary* of (x,y) with respect to the partition $\{R_0, R_1\}$ is the sequence $(a_i)_{i=\infty}^{+\infty} \in \Sigma$ given by

$$\begin{cases} a_k = 0 & \text{if } F^k(x,y) \in R_0, \\ a_k = 1 & \text{if } F^k(x,y) \in R_1 \end{cases}$$

In particular, if $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ is the *itineary* of $\mathcal{O}_F((x,y))$ we have

$$F^k(x,y) \in R_{a_k}, \quad \text{for all} \quad k \in \mathbb{Z}.$$

Note that here, since F is invertible, we can record not only the future but also the past.

Let us now define sets of points which share the same finite piece of itinerary. Given $n, m \in \mathbb{N}$ and $a_k \in \{0,1\}$ for $-m \le k \le n$, let

$$R_{-m,n}(a_{-m}, \ldots, a_n) = \{(x,y) \in X | \quad F^k(x,y) \in R_{a_k} \text{ for } -m \le k \le n\}.$$

These are all points such that the block of the itineary from $-m$ to n is given by the digits

$$a_{-m}, a_{-m+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-1}, a_n.$$

To construct such sets, let us rewrite them as

$$R_{-m,n}(a_{-m}, \ldots, a_n) = \bigcap_{k=-m}^{n} F^{-k}(R_{a_k}).$$

Example 1.4.1. *Let us compute $F^{-1}(R_0)$. Either from the definition or from the geometric action of F^{-1}, one can see that (see Figure 1.5(a))*

$$F^{-1}(R_0) = \left[0, \frac{1}{4}\right) \times [0,1) \cup \left[\frac{1}{2}, \frac{3}{4}\right) \times [0,1).$$

Thus, (see Figure 1.5(b))

$$R_{0,1}(1,0) = R_1 \cap F^{-1}(R_0) = \left[\frac{1}{2}, \frac{3}{4}\right) \times [0,1).$$

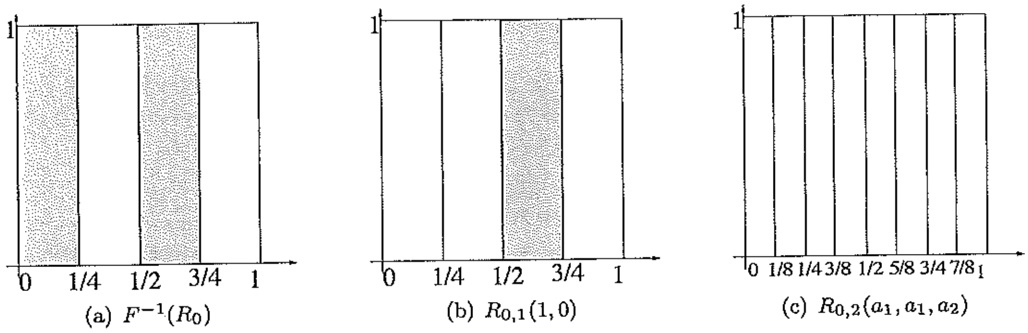

Figure 1.5: Examples of rectangles determined by future itineraries.

One can prove that all rectangles deterimed by forward itineraries, i.e. of the form $R_{0,n}(a_0, a_1, \dots, a_n)$, are thin vertical rectangles of width $1/2^{n+1}$ and full height, as in Figure 1.5(c), and as $a_0, \dots, a_n$ changes, they cover X. More precisely, recalling the intervals $I(a_0, \dots, a_n)$ defined for the doubling map[10], we have

$$R_{0,n}(a_0, a_1, \dots, a_n) = I(a_0, \dots, a_n) \times [0,1) = \left[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}\right) \times [0,1) \quad \text{for some } 0 \le k < 2^{n+1}.$$

Let us now describe a set which share the same *past* itinerary.

Example 1.4.2. *The image $F(R_0)$ is the bottom horizontal rectangle in the left square in Figure 1.4. The image $F^2(R_0)$ is shown in Figure 1.6(a) and is given by*

$$F^2(R_1) = [0,1) \times \left[0, \frac{1}{4}\right) \cup [0,1) \times \left[\frac{1}{2}, \frac{3}{4}\right).$$

[Try to convince yourself by imagining the geometric action of F on these sets (or by writing an explicit formula)]. Hence, for example (see Figure 1.6(b))

$$R_{-2,-1}(0,1) = F^2(R_0) \cap F(R_1) = [0,1) \times \left[\frac{1}{2}, \frac{3}{4}\right).$$

In general, one can verify that each set of the form

$$R_{-n,-1}(a_{-n}, \dots, a_{-1})$$

(dependend only on the past itinerary) is a thin horizontal rectangle, of height $1/2^n$ and full width, as in Figure 1.6(c).

Exercise 1.4.2. *Draw the following sets:*

(a) $R_{-1,0}(0,1)$

[10]This is because the *future* history of F, i.e. whether $F^k(x,y)$ with $k \ge 0$ belongs to R_0 or R_1, is completely deterimed by the doubling map.

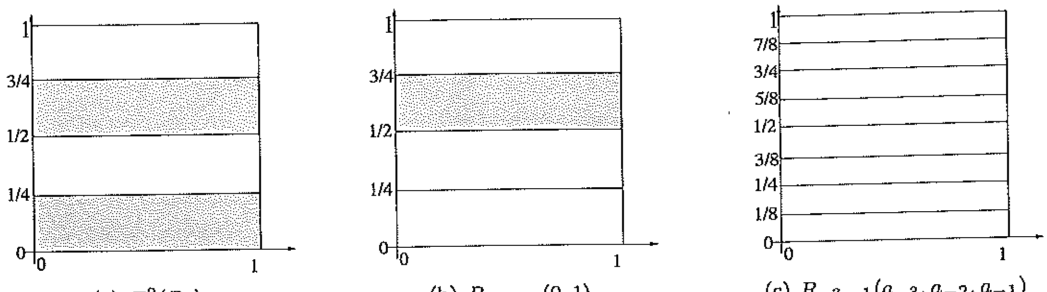

Figure 1.6: Examples of rectangles determined by past itineraries.

(b) $R_{-1,1}(0,1,1)$

(c) $R_{-2,0}(1,0,1)$

In general $R_{-m,n}(a_{-m},\dots,a_n)$ is a rectangle of horizontal width $1/2^{n+1}$ and height $1/2^m$.

The more we precise the backwards itinerary $a_{-1},a_{-2},\dots,a_{-n}$, the thinner the precision with which we determine the vertical component y. Moreover, from the geometric picture, you can guess that as $a_0,a_1,\dots,a_n,\dots$ give the digits of the binary expansion of x, $a_{-1},a_{-2},\dots,a_{-n},\dots$ give the digits of the binary expansion of y. This is exactly the insight that we need to construct the semi-conjugacy with the full shift.

Now we are ready to construct a semi-conjugacty beteween the baker map and the full shift which is a conjugacy outside a measure zero set of points.

Denote by
$$T_1 = \{\underline{a} \in \Sigma : \exists i_0 \in \mathbb{Z} \text{ such that } a_i = 1 \forall i > i_0\}$$
the set of sequences with forward tails consisting only of 1s. We have $\sigma(T_1) = T_1$, i.e., T_1 is a shift-invariant subspace. Hence its complement $\tilde{\Sigma} = \Sigma \setminus T_1$ is also shift-invariant.

Theorem 1.4.1. *The baker map is semi-conjugated to the full shift $\sigma : \tilde{\Sigma} \to \tilde{\Sigma}$ via the map $\Psi : \Sigma \to X$ given by*
$$\Psi((a_i)_{i=-\infty}^{+\infty}) = (x,y) \quad \textit{where} \quad x = \sum_{i=1}^{\infty} \frac{a_{i-1}}{2^i} \mod 1, \quad y = \sum_{i=1}^{\infty} \frac{a_{-i}}{2^i} \mod 1.$$

As for the doubling map, binary expansions turns out to be crucial to build the map Ψ. While the *future* $(a_i)_{i=0}^{\infty}$ of the sequence $(a_i)_{i=-\infty}^{\infty}$ will be used to give the binary expansion of x, the *past* $(a_i)_{i=-\infty}^{-1}$ of the sequence $(a_i)_{i=-\infty}^{\infty}$ turns out to be related to the binary expansion of the vertical coordinate y.

Proof of Theorem 1.4.1. For every point (x,y), both x and y can be expressed in binary expansion. If x has a binary expansion of the form $a_0,\dots,a_{i_0},0,1,1,1,1,1,\dots$ (i.e., with a forbidden tail), then x also has the binary expansion $a_0,\dots,a_{i_0},1,0,0,0,0,0,\dots$ (exercise!). This shows that Ψ is surjective.

Thus it remains to check that $\Psi\sigma = F\Psi$. Let us first compute
$$\Psi\left(\sigma((a_i)_{i=-\infty}^{+\infty})\right) = \Psi\left((a_{i+1})_{i=-\infty}^{+\infty}\right) = \left(\sum_{i=1}^{\infty} \frac{a_i}{2^i} \mod 1, \sum_{i=1}^{\infty} \frac{a_{-i+1}}{2^i} \mod 1\right).$$

Let a_0 be the first digit of the binary expansion of x. We have that $a_0 = 0$ if $0 \le x < \frac{1}{2}$ and $a_0 = 1$ if $\frac{1}{2} \le x < 1$. For $x = \frac{1}{2}$ we have either $a_0 = 1$ (then $a_i = 0$ for $i \ge 1$) or $a_0 = 0$ (then $a_i = 1$ for $i \ge 1$). The latter has a forbidden tail and thus does not occur. Therefore $a_0 = 0$ if $0 \le x < \frac{1}{2}$ and $a_0 = 1$ if $\frac{1}{2} \le x < 1$, which allows us to write
$$F(x,y) = \left(2x \mod 1, \frac{y + a_0}{2}\right).$$

