Handout 2

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„Classical Fourier Analysis

Chapter 3
Fourier Analysis on the Torus

Principles of Fourier series go back to ancient times. The attempts of the Pythagorean school to explain musical harmony in terms of whole numbers embrace early elements of a trigonometric nature. The theory of epicycles in the *Almagest* of Ptolemy, based on work related to the circles of Appolonius, contains ideas of astronomical periodicities that we would interpret today as harmonic analysis. Early studies of acoustical and optical phenomena, as well as periodic astronomical and geophysical occurrences, provided a stimulus of the physical sciences to the rigorous study of expansions of periodic functions. This study is carefully pursued in this chapter.

The modern theory of Fourier series begins with attempts to solve boundary value problems using trigonometric functions. The work of d'Alembert, Bernoulli, Euler, and Clairaut on the vibrating string led to the belief that it might be possible to represent arbitrary periodic functions as sums of sines and cosines. Fourier announced belief in this possibility in his solution of the problem of heat distribution in spatial bodies (in particular, for the cube $\mathbf{T}^3$) by expanding an arbitrary function of three variables as a triple sine series. Fourier's approach, although heuristic, was appealing and eventually attracted attention. It was carefully studied and further developed by many scientists, but most notably by Laplace and Dirichlet, who were the first to investigate the validity of the representation of a function in terms of its Fourier series. This is the main topic of study in this chapter.

3.1 Fourier Coefficients

We discuss some basic facts of Fourier analysis on the torus $\mathbf{T}^n$. Throughout this chapter, n denotes the dimension, i.e., a fixed positive integer.

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3.1.1 The n-Torus $\mathbf{T}^n$

The n-torus $\mathbf{T}^n$ is the cube $[0,1]^n$ with opposite sides identified. This means that the points $(x_1,\dots,0,\dots,x_n)$ and $(x_1,\dots,1,\dots,x_n)$ are identified whenever 0 and 1 appear in the same coordinate. A more precise definition can be given as follows: For x,y in $\mathbf{R}^n$, we say that

$$x \equiv y \tag{3.1.1}$$

if $x-y \in \mathbf{Z}^n$. Here $\mathbf{Z}^n$ is the additive subgroup of all points in $\mathbf{R}^n$ with integer coordinates. If (3.1.1) holds, then we write $x = y \pmod{\mathbf{1}}$. It is a simple fact that $\equiv$ is an equivalence relation that partitions $\mathbf{R}^n$ into equivalence classes. The n-torus $\mathbf{T}^n$ is then defined as the set $\mathbf{R}^n/\mathbf{Z}^n$ of all such equivalence classes. When $n=1$, this set can be geometrically viewed as a circle by bending the line segment $[0,1]$ so that its endpoints are brought together. When $n=2$, the identification brings together the left and right sides of the unit square $[0,1]^2$ and then the top and bottom sides as well. The resulting figure is a two-dimensional manifold embedded in $\mathbf{R}^3$ that looks like a donut. See Figure 3.1.

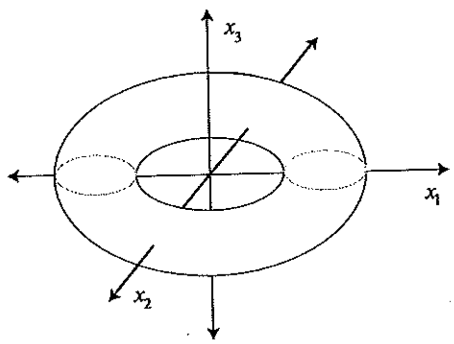

Fig. 3.1 The graph of the two-dimensional torus $\mathbf{T}^2$.

The n-torus is an additive group, and zero is the identity element of the group, which of course coincides with every $e_j = (0,\dots,0,1,0,\dots,0)$. To avoid multiple appearances of the identity element in the group, we often think of the n-torus as the set $[-1/2,1/2]^n$. Since the group $\mathbf{T}^n$ is additive, the inverse of an element $x \in \mathbf{T}^n$ is denoted by $-x$. For example, $-(1/3,1/4) \equiv (2/3,3/4)$ on $\mathbf{T}^2$, or, equivalently, $-(1/3,1/4) = (2/3,3/4) \pmod{\mathbf{1}}$.

The n-torus $\mathbf{T}^n$ can also be thought of as the following subset of $\mathbf{C}^n$,

$$\{(e^{2\pi i x_1},\dots,e^{2\pi i x_n}) \in \mathbf{C}^n \,:\, (x_1,\dots,x_n) \in [0,1]^n\}, \tag{3.1.2}$$

in a way analogous to which the unit interval $[0,1]$ can be thought of as the unit circle in $\mathbf{C}$ once 1 and 0 are identified.

Functions on $\mathbf{T}^n$ are functions f on $\mathbf{R}^n$ that satisfy $f(x+m) = f(x)$ for all $x \in \mathbf{R}^n$ and $m \in \mathbf{Z}^n$. Such functions are called 1-*periodic* in every coordinate. Haar measure on the n-torus is the restriction of n-dimensional Lebesgue measure to the set $\mathbf{T}^n = [0,1]^n$. This measure is still denoted by dx, while the measure of a set $A \subseteq \mathbf{T}^n$ is denoted by $|A|$. Translation invariance of the Lebesgue measure and the periodicity of functions on $\mathbf{T}^n$ imply that for all f on $\mathbf{T}^n$, we have

$$\int_{\mathbf{T}^n} f(x)\,dx = \int_{[-1/2,1/2]^n} f(x)\,dx = \int_{[a_1,1+a_1]\times\cdots\times[a_n,1+a_n]} f(x)\,dx \tag{3.1.3}$$

for any real numbers $a_1,\dots,a_n$. The L^p spaces on $\mathbf{T}^n$ are nested and L^1 contains all L^p spaces for $p \geq 1$.

Elements of $\mathbf{Z}^n$ are denoted by $m = (m_1,\dots,m_n)$. For $m \in \mathbf{Z}^n$, we define the *total size* of m to be the number $|m| = (m_1^2+\cdots+m_n^2)^{1/2}$. Recall that for $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$ in $\mathbf{R}^n$,

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n$$

denotes the usual dot product. Finally, for $x \in \mathbf{T}^n$, $|x|$ denotes the usual Euclidean norm of x. If we identify $\mathbf{T}^n$ with $[-1/2,1/2]^n$, then $|x|$ can be interpreted as the distance of the element x from the origin, and then we have that $0 \leq |x| \leq \sqrt{n}/2$ for all $x \in \mathbf{T}^n$.

3.1.2 Fourier Coefficients

Definition 3.1.1. For a complex-valued function f in $L^1(\mathbf{T}^n)$ and m in $\mathbf{Z}^n$, we define

$$\widehat{f}(m) = \int_{\mathbf{T}^n} f(x) e^{-2\pi i m\cdot x}\,dx. \tag{3.1.4}$$

We call $\widehat{f}(m)$ the mth *Fourier coefficient* of f. We note that $\widehat{f}(m)$ is not defined for $\xi \in \mathbf{R}^n \setminus \mathbf{Z}^n$, since the function $x \mapsto e^{-2\pi i \xi\cdot x}$ is not 1-periodic in every coordinate and therefore not well defined on $\mathbf{T}^n$.

The *Fourier series* of f at $x \in \mathbf{T}^n$ is the series

$$\sum_{m\in\mathbf{Z}^n} \widehat{f}(m) e^{2\pi i m\cdot x}. \tag{3.1.5}$$

It is not clear at present in which sense and for which $x \in \mathbf{T}^n$ (3.1.5) converges. The study of convergence of Fourier series is the main topic of study in this chapter.

We quickly recall the notation we introduced in Chapter 2. We denote by $\overline{f}$ the complex conjugate of the function f, by $\widetilde{f}$ the function $\widetilde{f}(x) = f(-x)$, and by $\tau^y(f)$ the function $\tau^y(f)(x) = f(x-y)$ for all $y \in \mathbf{T}^n$. We mention some elementary properties of Fourier coefficients.

Proposition 3.1.2. *Let f, g be in $L^1(\mathbf{T}^n)$. Then for all $m,k \in \mathbf{Z}^n$, $\lambda \in \mathbf{C}$, $y \in \mathbf{T}^n$, and all multi-indices α we have*

(1) $\widehat{f+g}(m) = \widehat{f}(m) + \widehat{g}(m)$,

(2) $\widehat{\lambda f}(m) = \lambda \widehat{f}(m)$,

(3) $\widehat{\overline{f}}(m) = \overline{\widehat{f}(-m)}$,

(4) $\widehat{\widetilde{f}}(m) = \widehat{f}(-m)$,

(5) $\widehat{\tau^y(f)}(m) = \widehat{f}(m)e^{-2\pi i m\cdot y}$,

(6) $(e^{2\pi i k(\cdot)}f)^\wedge(m) = \widehat{f}(m-k)$,

(7) $\widehat{f}(0) = \int_{\mathbf{T}^n} f(x)\,dx$,

(8) $\sup_{m\in\mathbf{Z}^n} |\widehat{f}(m)| \le \|f\|_{L^1(\mathbf{T}^n)}$,

(9) $\widehat{f*g}(m) = \widehat{f}(m)\widehat{g}(m)$,

(10) $\widehat{\partial^\alpha f}(m) = (2\pi i m)^\alpha \widehat{f}(m)$, *whenever* $f \in \mathscr{C}^\alpha$.

Proof. The proof of Proposition 3.1.2 is obvious and is left to the reader. We only sketch the proof of (9). We have

$$\widehat{f*g}(m) = \int_{\mathbf{T}^n}\int_{\mathbf{T}^n} f(x-y)g(y)e^{-2\pi i m\cdot(x-y)}e^{-2\pi i m\cdot y}\,dy\,dx = \widehat{f}(m)\widehat{g}(m),$$

where the interchange of integrals is justified by the absolute convergence of the integrals and Fubini's theorem. □

Remark 3.1.3. The Fourier coefficients have the following property. For a function f_1 on $\mathbf{T}^{n_1}$ and a function f_2 on $\mathbf{T}^{n_2}$, the tensor function

$$(f_1 \otimes f_2)(x_1,x_2) = f_1(x_1)f_2(x_2)$$

is a periodic function on $\mathbf{T}^{n_1+n_2}$ whose Fourier coefficients are

$$\widehat{f_1 \otimes f_2}(m_1,m_2) = \widehat{f_1}(m_1)\widehat{f_2}(m_2), \tag{3.1.6}$$

for all $m_1 \in \mathbf{Z}^{n_1}$ and $m_2 \in \mathbf{Z}^{n_2}$.

Definition 3.1.4. A *trigonometric polynomial* on $\mathbf{T}^n$ is a function of the form

$$P(x) = \sum_{m\in\mathbf{Z}^n} a_m e^{2\pi i m\cdot x}, \tag{3.1.7}$$

where $\{a_m\}_{m\in\mathbf{Z}^n}$ is a finitely supported sequence in $\mathbf{Z}^n$. The *degree* of P is the largest number $|q_1|+\cdots+|q_n|$ such that a_q is nonzero, where $q=(q_1,\dots,q_n)$.

Example 3.1.5. A *trigonometric monomial* is a function of the form

$$P(x) = a\, e^{2\pi i(q_1x_1+\cdots+q_nx_n)}$$

for some $q=(q_1,\dots,q_n)\in\mathbf{Z}^n$ and $a\in\mathbf{C}$. Observe that $\widehat{P}(q)=a$ and $\widehat{P}(m)=0$ for $m\neq q$.

Let $P(x)=\sum_{|m|\le N} a_m e^{2\pi i m\cdot x}$ be a trigonometric polynomial and let f be in $L^1(\mathbf{T}^n)$. Exercise 3.1.1 gives that $(f*P)(x)=\sum_{|m|\le N} a_m\widehat{f}(m)e^{2\pi i m\cdot x}$. This implies that the partial sums $\sum_{|m|\le N}\widehat{f}(m)e^{2\pi i m\cdot x}$ of the Fourier series of f given in (3.1.5) can be obtained by convolving f with the functions

$$D_N(x) = \sum_{|m|\le N} e^{2\pi i m\cdot x}. \tag{3.1.8}$$

These expressions are named after Dirichlet, as the following definition indicates.