SF 2720 -Homework assignment 2 - due Tuesday 27. 9. 2016.

1. Suppose that $f : X \to X$ and $g : Y \to Y$ are topologically semi-conjugate via a continuous onto map $\phi : X \to Y$. We say $g : Y \to Y$ is a *topological factor* of $f : X \to X$.

   a) Prove that if $f : X \to X$ is topologically transitive then $g : Y \to Y$ is topologically transitive. Show that the converse implication does not hold by providing a counterexample.

   b) Prove that if $f : X \to X$ is topologically mixing then $g : Y \to Y$ is topologically mixing. Show that the converse implication does not hold by providing a counterexample.

2. For $\alpha \in \mathbb{R}$, the map F_α (often called the skew-shift map) of the 2 dimensional torus $\mathbb{T}^2$ is:

   $$F_\alpha : (x, y) \mapsto (x + \alpha \mod 1, x + y \mod 1).$$

   Observe that $R_\alpha : x \mapsto x+\alpha \mod 1$, is a topological factor of F_α via the map $\phi : (x, y) \mapsto x$.

   a) Find necessary and sufficient condition on α which guarantees that F_α is topologically transitive. Is F_α minimal?

   b) Is F_α topologically mixing?

   c) Is F_α an isometry (i.e. does F preserve the distance on the torus which is induced by the Eucledian distance in $\mathbb{R}^2$)?

3. Consider the map $G : [0, 1)^2 \to [0, 1)^2$ defined by: $F(x, y) = (3x, \frac{y}{3})$ for $0 \le x < \frac{1}{3}$, $F(x, y) = (3x - 1, \frac{y}{3} + \frac{1}{3})$ for $\frac{1}{3} \le x < \frac{2}{3}$, and $F(x, y) = (3x - 2, \frac{y}{3} + \frac{2}{3})$ for $\frac{2}{3} \le x < 1$.

   a) Draw a picture to show how this map acts on the square.

   b) Show that G is semi-conjugated to the shift $\sigma : \Sigma_3 \to \Sigma_3$ on the space Σ_3 of doubly infinite sequences on 3 symbols $\{0, 1, 2\}$. (Hint: read Handout 5 from 'Readings' section.)

   c) Give an example of a periodic point of period 5 for G.

4. An integer 2 by 2 matrix A of determinant one induces a map on the 2-torus $\mathbb{T}^2$ which we denote by F_A. The map F_A is called a *toral automorphism.* F_A is called *hyperbolic* toral automorphism if for every eigenvalue λ of A we have $|\lambda| \neq 1$. Consider the following matrices with integer entries:

   $$a)\ A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad b)\ A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad c)\ A_3 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad d)\ A_4 = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

   a) For each of the matrices above say if the corresponding map F_{A_i} $(i = 1, 2, 3, 4)$ is:

   - a toral automorphism? Justify your answer.

   -a hyperbolic toral automorphism? Justify your answer.

   b) Draw geometrically (as we did in class for the CAT map) how A_1 and A_4 transform the unit square. Prove or disprove the mixing property for F_{A_1}, and for F_{A_4}.

   c) In the class we called a dynamical system chaotic if it has a dense set of periodic points, if it is topologically transitive and has a sensitive dependence on initial conditions. Is either of the two maps F_{A_1} and F_{A_4} a chaotic map on the 2-torus?

5. a) Exercise A.5.4. from Handout 4. (All the handouts are posted in the 'Readings' section).

   b) Exercise A.5.7. from Handout 4.