Def: ((signless) stirling number of the first kind) 13

$$c(n,k) = |\{ \pi \in \mathfrak{S}_n : c(\pi) = k \}|$$

Prop: $$\sum_{k=0}^{n} c(n,k) t^k = t(t+1)\cdots(t+n-1) \qquad (*)$$

Proof: See book for combinatorial proof.

Let $F_n(t)$ be LHS of $(*)$.

Then $\frac{F_n(t)}{n!} = Z_n(t,t,t,\ldots)$, and hence

$$\sum_{n=0}^{\infty} \frac{F_n(t)}{n!} x^n = \exp\left(\sum_{k=1}^{\infty} t \frac{x^k}{k} \right) = \exp(-t \log(1-x))$$

$$= (1-x)^{-t} = \sum_{n=0}^{\infty} (-1)^n \binom{-t}{n} x^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(-t)(-t-1)\cdots(-t-k+1)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{t(t+1)\cdots(t+n-1)}{n!} x^n$$

$\square$

Corollary: $c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1)$, $n,k \geq 1$

Prf: Use prop or: Take $\pi \in \mathfrak{S}_n$. Either

(1). $\pi(n) = n$ (own cycle). Counted by $c(n-1,k-1)$

(2). $\pi(n) \neq n$. Insert n infront of any $i \in [n-1]$ in a permution in $\mathfrak{S}_{n-1}$ with k cycles. Counted by $(n-1)c(n-1,k)$.

Inversions : Define $I : \mathfrak{S}_n \to [0,n-1]\times[0,n-2]\times\ldots\times[0,0]$

If $\pi = a_1 a_2 \ldots a_n \in \mathfrak{S}_n$, let $I(\pi) = (b_1, \ldots, b_n)$ $\quad$ ($[a,b] = \{a, a+1, \ldots, b\}$)

where $b_s = \#$ t's s.t. $t > s$ and t is to the left of s $\quad$ 14

Hence $b_n = 0$, $b_{n-1} \le 1$, ..., $b_k \le n-k$, ..., $b_1 \le n-1$

$I(41532) = (1,3,2,0,0)$

Inverse of I:

5		
45	since	$b_4 = 0$
453	"	$b_3 = 2$
4532	"	$b_4 = 3$
41532	"	$b_1 = 1$

Prop.
I is a bijection.

- An inversion in π is a pair (a_i, a_j) s.t. $i<j$ and $a_j > a_i$.
- let $inv(\pi)$ be the # of inversions in π.

$\therefore$ $inv(\pi) = b_1 + b_2 + \ldots + b_n$, $I(\pi) = (b_1, \ldots, b_n)$

Corollary (Rodriguez, 1839)

$$\sum_{\pi \in \mathfrak{S}_n} q^{inv(\pi)} = (1+q)(1+q+q^2)\ldots(1+q+\ldots+q^{n-1}) \qquad (\star)$$

Proof:
$$\sum_{\pi\in\mathfrak{S}_n} q^{inv(\pi)} = \sum_{b_1=0}^{n-1}\sum_{b_2=0}^{n-2}\ldots\sum_{b_n=0}^{0} q^{b_1} q^{b_2}\ldots q^{b_n} =$$

$$= (1+q+\ldots+q^{n-1})\ldots(1+q)(1). \quad \square$$

$(\star)$ is called "the q-analog of $n!$" and is denoted by $n!_q$. Denote by $(n)_q = 1+q+\ldots+q^{n-1} = \frac{q^n-1}{q-1}$. Then

$n!_q = (1)_q \ldots (n)_q$

Descents

- A <u>descent</u> in a permutation $\pi = a_1 a_2 \dots a_n \in \mathfrak{S}_n$ is an i s.t. $a_i > a_{i+1}$.
- The <u>descent set</u> is $D(\pi) = \{i : a_i > a_{i+1}\} \subseteq [n-1]$

$\pi = 3\,5\,4\,2\,1 = \{2,3,4\}$

- The <u>descent statistic</u> is $des(\pi) = |D(\pi)|$.

<u>Define</u>:

$$\alpha(S) = \#\{\pi \in \mathfrak{S}_n : D(\pi) \subseteq S\}$$
$$\beta(S) = \#\{\pi \in \mathfrak{S}_n : D(\pi) = S\}$$

Hence
$$\alpha(S) = \sum_{T \subseteq S} \beta(T), \qquad (**)$$

and we shall see later that $(**)$ implies
$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha(T)$$

<u>Prop</u>. Let $S = \{s_1 < s_2 < \dots < s_k\} \subseteq [n-1]$. Then
$$\alpha(S) = \binom{n}{s_1, s_2 - s_1, s_3 - s_2, \dots, n - s_k}$$

<u>Proof</u>: $D(\pi) \subseteq S$ iff π may be written as
$\pi = \underset{s_1}{w_1} \underset{s_2}{w_2} \dots \underset{s_k}{w_k} w_{k+1}$ where w_i is an increasing word

$$w_1 = a_1 < a_2 < \dots < a_{s_1}$$
$$w_2 = a_{s_1+1} < \dots < a_{s_2}$$
$$\vdots$$
$$w_{k+1} = a_{s_k+1} < \dots < a_n$$

There are $\binom{n}{s_1}$ choices for w_1

$\binom{n-s_1}{s_2-s_1}$ —"— w_2

$\binom{n-s_2}{s_3-s_2}$ —"— w_3

we obtain

$$\alpha(S) = \binom{n}{s_1}\binom{n-s_1}{s_2-s_1}\cdots\binom{n-s_k}{n-s_k} = \binom{n}{s_1,\, s_2-s_1,\, \ldots,\, n-s_k}$$

$\square$

<u>Example</u>: $n \geq 8$

$$\begin{aligned}\beta_n(4,7) &= \alpha_n(4,7) - \alpha_n(4) - \alpha_n(7) + \alpha_n(\emptyset) = \\ &= \binom{n}{4,3,n-7} - \binom{n}{4} - \binom{n}{7} + 1 \\ &= \frac{n!}{4!\,3!\,(n-7)!}\,\frac{7!}{7!} - \binom{n}{4} - \binom{n}{7} + 1 = \\ &= \binom{7}{3}\binom{n}{7} - \binom{n}{4} - \binom{n}{7} + 1 \\ &= 34\binom{n}{7} - \binom{n}{4} + 1\end{aligned}$$

$\frac{7\cdot 6\cdot 5}{3\cdot 2} = 35$

The n^{th} <u>Eulerian polynomial</u>, $A_n(x)$, is

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}(\pi)+1} = \sum_{k=1}^{n} A(n,k)\, x^k,$$

where $A(n,k) = \#\{\pi \in \mathfrak{S}_n : \mathrm{des}(\pi) = k-1\}$.

$$\begin{aligned}A_0(x) &:= 1 \\ A_1(x) &= x \\ A_2(x) &= x + x^2 \\ A_3(x) &= x + 4x^2 + x^3 \\ A_4(x) &= x + 11x^2 + 11x^3 + x^4 \\ A_5(x) &= x + 26x^2 + 66x^3 + 26x^4 + x^5\end{aligned}$$

We say that des has the <u>Eulerian distribution</u>.

Note that $A(n,k) = A(n, n+1-k)$, that is des and $n-1-$des have the same distribution

This is clear because $n-1-\mathrm{des}(\pi) = |\{i : a_i < a_{i+1}\}|$
$= \mathrm{des}(\pi^r)$, where

$\pi^r = a_n a_{n-1} \ldots a_1$

An <u>excedance</u> in π is an i s.t. $\pi(i) > i$.

$exc(\pi) = |\{i : \pi(i) > i\}|$.

Recall the definition of $\hat{\pi}$

standard rep of $\pi = (54)(6)(713)(82)$

$\hat{\pi} = 54\ 6\ 713\ 82$

we see that $des(\hat{\pi}) = |\{i : i > \pi(i)\}| =$

$= |\{\pi(i) : \pi^{-1}(\pi(i)) > \pi(i)\}| = exc(\pi^{-1})$

Hence des and exc have the same distribution.

Note also that if $wexc(\pi) := |\{i : \pi(i) \geq i\}|$,

then $exc(\pi^{-1}) + wexc(\pi) = n$ i.e.,

$wexc(\pi) - 1 = n - 1 - exc(\pi^{-1})$

By the symmetry property of the Eulerian distr.

we have

<u>Prop</u>: des, exc, and $(wexc - 1)$ have the same distribution.

<u>Prop</u>: $A(n+1, k) = kA(n,k) + (n-k+2)A(n,k-1)$

<u>Proof</u>: Write $A(n+1,k) = |B(n,k)| + |C(n,k)|$

$B(n,k) = \{\pi \in \mathfrak{S}_{n+1} : des(\pi) = k-1,\ n+1$ is in a descent or last $\}$

$C(n,k) = \{\pi \in \mathfrak{S}_{n+1} : des(\pi) = k-1,\ n+1$ is first, or not in a descent $\}$

$|B(n,k)| = kA(n,k)$

$|C(n,k)| = (n+1-1-(k-1)+1)\,A(n,k-1)$

$\square$

Prop: $\displaystyle\sum_{m=0}^{\infty} m^n x^m = \frac{A_n(x)}{(1-x)^{n+1}}$, for all $n \geq 0$. (18)

Proof: $n=0$ $\quad \sum x^m = \frac{1}{1-x}$ o.k

Induction: suppose true for n. Differentiate both sides, and multiply by x.

$$\sum_{m=0}^{\infty} m^{n+1} x^m = x \, \frac{A_n'(1-x)^{n+1} + A_n \cdot (n+1)(1-x)^n}{(1-x)^{2n+2}} =$$

$$= \frac{A_n' \cdot (1-x)x + (n+1)x A_n}{(1-x)^{n+2}} = \frac{(\dagger)}{(1-x)^{n+2}}$$

$$(\dagger) = \sum_k (n+1) A(n,k) x^{k+1} + \sum k A(n,k) x^k - \sum k A(n,k) x^{k+1}$$

$$= \sum_k (n+1) A(n,k-1) - (k-1) A(n,k-1) + k A(n,k) \quad x^k$$

$$= \sum_k (n+2-k) A(n,k-1) + k A(n,k) \, x^k$$

$$= \sum A(n+1,k) \, x^k$$