4.1

$X(t)$ and $Y(t)$ independent and WSS with ACFs:

$$\left.\begin{aligned} r_x(\tau) &= \sigma_x^2 e^{-a|\tau|} + m_x^2 \\ r_y(\tau) &= \sigma_y^2 e^{-b|\tau|} + m_y^2 \end{aligned}\right\} \text{ we define } Z(t) = X(t)Y(t)$$

• $r_z(t,\tau) = E[X(t+\tau)Y(t+\tau)X(t)Y(t)] = \left\{\begin{array}{c}\text{independence}\\ X(t) \text{ and } Y(t)\end{array}\right\} =$

$= E[X(t+\tau)X(t)]\,E[Y(t+\tau)Y(t)] = \left\{\begin{array}{c}Y(t) \text{ and } X(t) \text{ are}\\ \text{WSS}\end{array}\right\} =$

$= r_x(\tau)\, r_y(\tau).$

• $R_z(f) = \mathcal{F}\{r_z(\tau)\} = R_x(f) * R_y(f).$

Therefore

$$r_z(\tau) = \left(\sigma_x^2 e^{-a|\tau|} + m_x^2\right)\left(\sigma_y^2 e^{-b|\tau|} + m_y^2\right)$$

we don't want to calculate the convolution of the two fourier transform. Hence, we will express $r_z(\tau)$ in a way that we can easily compute its Fourier transform.

$$r_z(\tau) = \sigma_x^2\sigma_y^2 e^{-(a+b)|\tau|} + m_x^2 m_y^2 + m_x^2\sigma_y^2 e^{-b|\tau|} + m_y^2\sigma_x^2 e^{-a|\tau|}$$

$\Rightarrow$ The Fourier transform of $e^{-a|t|} \rightarrow \dfrac{2a}{a^2 + (2\pi f)^2}$

$\Rightarrow$ The Fourier transform of a constant $A \rightarrow A\,\delta(f)$

$$R_z(f) = \sigma_x^2\sigma_y^2 \frac{2(a+b)}{(a+b)^2 + (2\pi f)^2} + m_x^2 m_y\, \delta(f) + m_x^2\sigma_y^2 \frac{2b}{b^2 + (2\pi f)^2} +$$

$$+ m_y^2\sigma_x^2 \frac{2a}{a^2 + (2\pi f)^2}$$

4.8|

AM signal $s(t) = c_0 A(t) \cos(2\pi f_c t + \bar{\phi})$

$\bar{\phi}$ R.V. uniform $[0, 2\pi]$
$A(t)$. stationary process $r_A(\tau)$, $R_A(f)$ } independent

a) Show $s(t)$ is WSS
b) Obtain the mean of $s(t)$

- $E[S(t)] = E[c_0 A(t) \cos(2\pi f_c t + \bar{\phi})] = c_0 E[A(t) \cos(2\pi f_c t + \bar{\phi})] =$

$$= \left\{ \begin{aligned} & E[Y g(X)] = \iint g(x) \cdot y \, f_{XY}(x,y)\, dx\, dy = \{\text{independence}\} = \\ & = \iint g(x) y \, f_X(x) f_Y(y)\, dx\, dy = \int g(x) f_X(x)\, dx \int y f_Y(y)\, dy = \\ & = E[Y]\, E[g(X)] \end{aligned} \right\} =$$

$$= c_0 E[A(t)]\, E[\cos(2\pi f_c t + \bar{\phi})] = m_A \cdot \int_0^{2\pi} \cos(2\pi f_c t + \bar{\phi}) \cdot \frac{1}{2\pi}\, d\bar{\phi} =$$

$$= \frac{m_A}{2\pi} \sin(2\pi f_c + \phi)\Big]_0^{2\pi} = \frac{m_a}{2\pi}(\sin(2\pi f_c) - \sin(2\pi f_c)) = \emptyset$$

- $E[S(t+\tau) S(t)] = c_0^2 E[A(t+\tau) \cos(2\pi f_c(t+\tau) + \bar{\phi}) A(t) \cos(2\pi f_c t)] =$

$= \{\text{independence } A(t) \text{ and } \phi\} = c_0^2 E[A(t+\tau) A(t)]\, E[\cos(2\pi f_c(t+\tau) + \phi) \cdot$

$\cdot \cos(2\pi f_c t)] = c_0^2 r_A(\tau)\, E[\frac{1}{2}\cos(2\pi f_c(2t+\tau) + 2\bar{\phi}) + \frac{1}{2}\cos(2\pi f_c \tau)] =$

$$= r_A(\tau)\left(\int_0^{2\pi} \frac{1}{2}\cos(2\pi f_c(2t+\tau) + 2\bar{\phi}) \frac{1}{2\pi}\, d\phi + \frac{1}{2}\cos(2\pi f_c \tau)\right) c_0^2 =$$

$$= c_0^2 r_A(\tau)\left(\frac{1}{8\pi} \sin(2\pi f_c(2t+\tau) + 2\bar{\phi})\Big]_0^{2\pi} + \frac{1}{2}\cos(2\pi f_c \tau)\right) =$$

$$= c_0^2 r_A(\tau)\left(\emptyset + \frac{1}{2}\cos(2\pi f_c \tau)\right)$$

$s(t)$ is stationary

c) $R_S(f)$ of $s(t)$.

$$r_S(\tau) = r_A(\tau)\left(\frac{1}{2}\cos(2\pi f_c \tau)\right) c_0^2$$

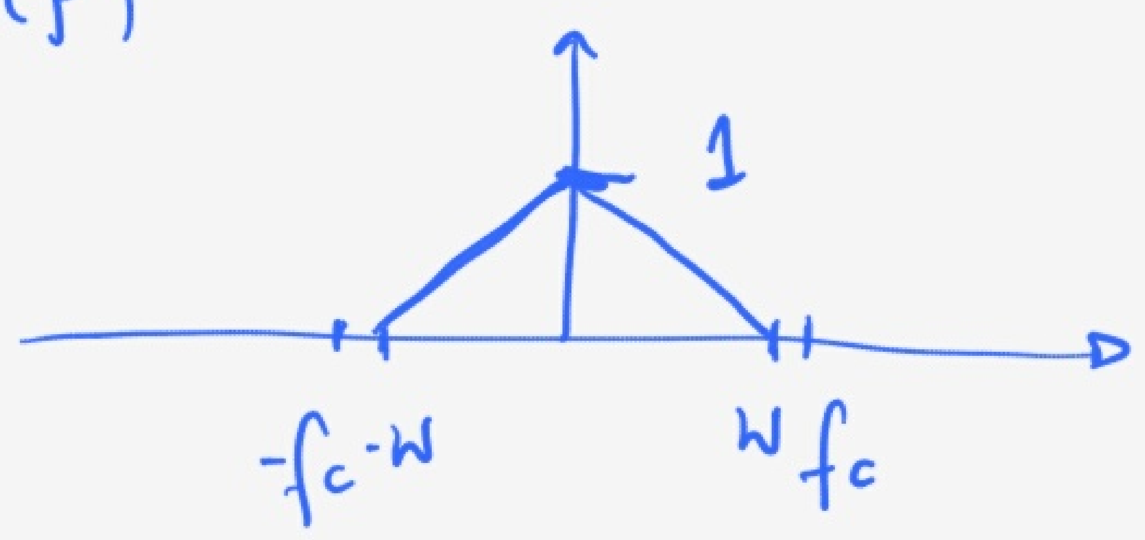
$$R_S(f) = R_A(f) * \left(\frac{1}{4}\delta(f - f_c) + \frac{1}{4}\delta(f + f_c)\right) c_0^2$$

$$R_S(f) = \frac{1}{4}\left(R_A(f - f_c) + R_A(f + f_c)\right) c_0^2$$

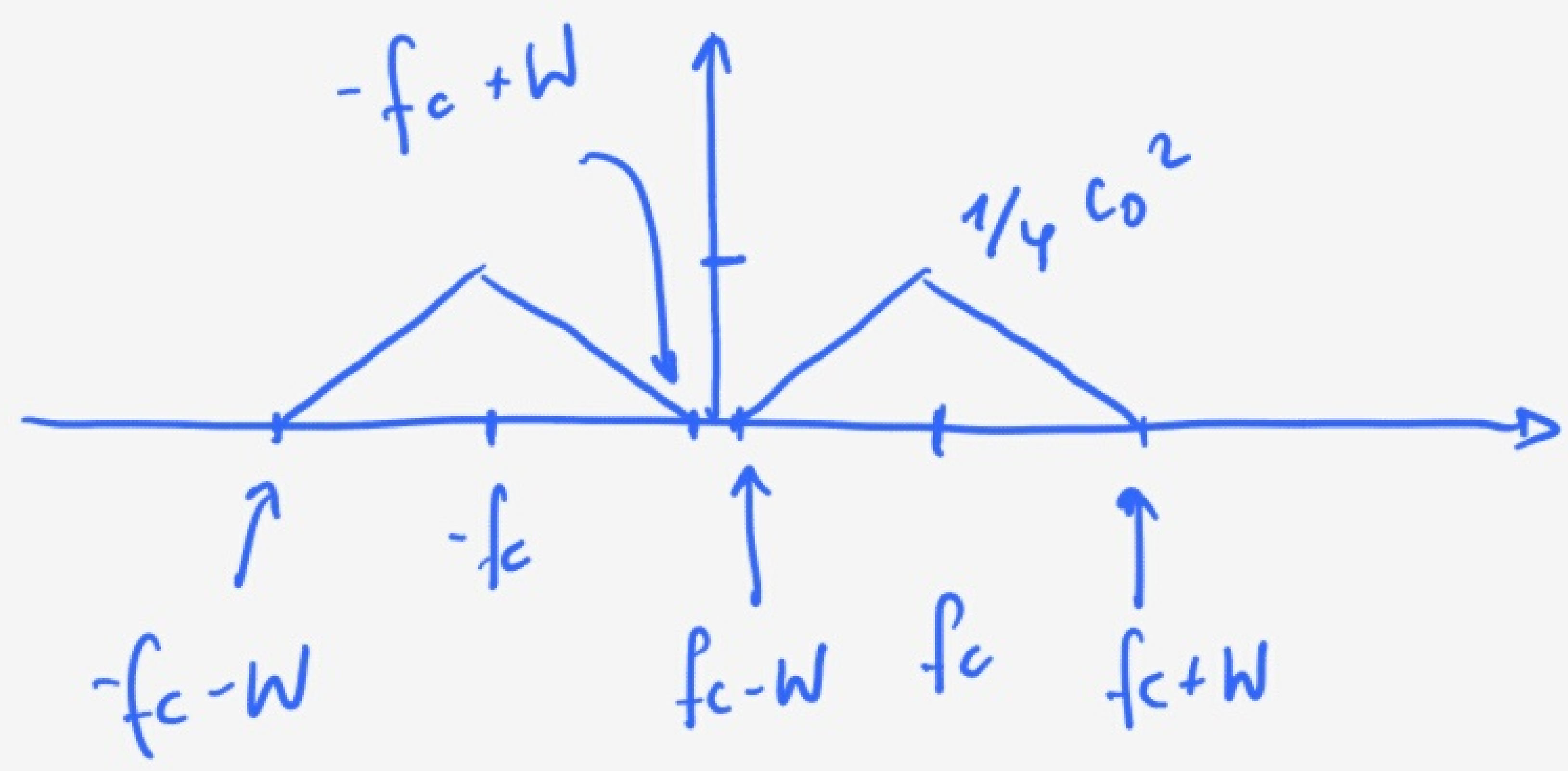
$R_S(f) = \frac{1}{4}\left(R_A(f+f_c) + R_A(f-f_c)\right) c_0^2$

$R_A(f) = 0$ when $|f| > W$, $W < f_c$

Example of $R_A(f)$



Then $R_S(f)$



4.9] Determine $R(\nu)$ of

b) $r(k) = \delta(k) + \frac{1}{|k|!}$

$$R(\nu) = \sum_{-\infty}^{\infty}\left(\delta(k) + \frac{1}{|k|!}\right) e^{-j2\pi k\nu} = \sum_{-\infty}^{-1} \frac{1}{|k|!} e^{-j2\pi k\nu} + \left(\delta(0) + \frac{1}{0!}\right) e^{\emptyset}$$

$$+ \sum_{1}^{\infty} \frac{1}{|k|!} e^{j2\pi k\nu} = \left\{ \sum_{-\infty}^{-1} \frac{1}{|k|!} e^{-j2\pi k\nu} = \sum_{1}^{\infty} \frac{1}{k!} e^{j2\pi k\nu} \right\}$$

0! is 1 convention

$$= \sum_{1}^{\infty} \frac{1}{k!}\left(e^{j2\pi k\nu} + e^{-j2\pi k\nu}\right) + 2 =$$

$$= \sum_{1}^{\infty} \frac{1}{k!}\left(\left(e^{j2\pi\nu}\right)^k + \left(e^{-j2\pi\nu}\right)^k\right) + 2 =$$

$= \{$ Maclaurin's serier $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (Taylor extension for complex space) $\}$

$$= e^{e^{j2\pi\nu}} + e^{e^{-j2\pi\nu}} = e^{\cos(2\pi\nu) + j\sin(2\pi\nu)} + e^{\cos(2\pi\nu) - j\sin(2\pi\nu)} =$$

$$= e^{\cos(2\pi\nu)} e^{+j\sin(2\pi\nu)} + e^{\cos(2\pi\nu)} e^{-j\sin(2\pi\nu)} =$$

$$= e^{\cos(2\pi\nu)} \left(e^{j\sin(2\pi\nu)} + e^{-j\sin(2\pi\nu)} \right) =$$

$$= e^{\cos(2\pi\nu)} \left(2\cos\left(\sin(2\pi\nu)\right)\right) \rfloor$$

4.10] ACF of process with spectral density:

b) $$R(\nu) = \frac{9}{10 - 6\cos(2\pi\nu)} = \left\{ DFT\left(a^{|n|}\right)(\nu) = \frac{1-a^2}{1+a^2-2a\cos(2\pi\nu)} \right\}$$

We need to find $\underline{a}$ and $\underline{c}$ so that:

$$\frac{9}{10 - 6\cos(2\pi\nu)} = \frac{c(1-a^2)}{b(1+a^2-2a\cos(2\pi\nu))}$$

For the denominator to be equal:

$$9 = c(1-a^2) \rightarrow$$

$$10 = b(1+a^2)$$

$$-6\cos(2\pi\nu) = -2ab\cos(2\pi\nu) \Rightarrow -6 = -2ab$$

$$3 = ab \qquad b = 3/a.$$

$$10 = b(1+a^2) = \frac{3}{a}(1+a^2)$$

$$10a = 3(1+a^2) \rightarrow 3a^2 - 10a + 3 = 0$$

$$a = \frac{10 \pm \sqrt{(100) - 36}}{6} = \frac{10 \pm 8}{6} \nearrow \begin{matrix} 2/6 = 1/3 \rfloor \\ 18/6 = 3 \rfloor \end{matrix}$$

two possible values for a will lead to other two possible values of b and c (one for each value of a).

$\Rightarrow$ Note that we want the function to be an acf function! If we select $a = 3$ the function is increasing with k which implies it cannot be an ACF! Hence, we must choose $a = 1/3$.

If $a = 1/3$ we have that $3 = ab \Rightarrow b = 9$

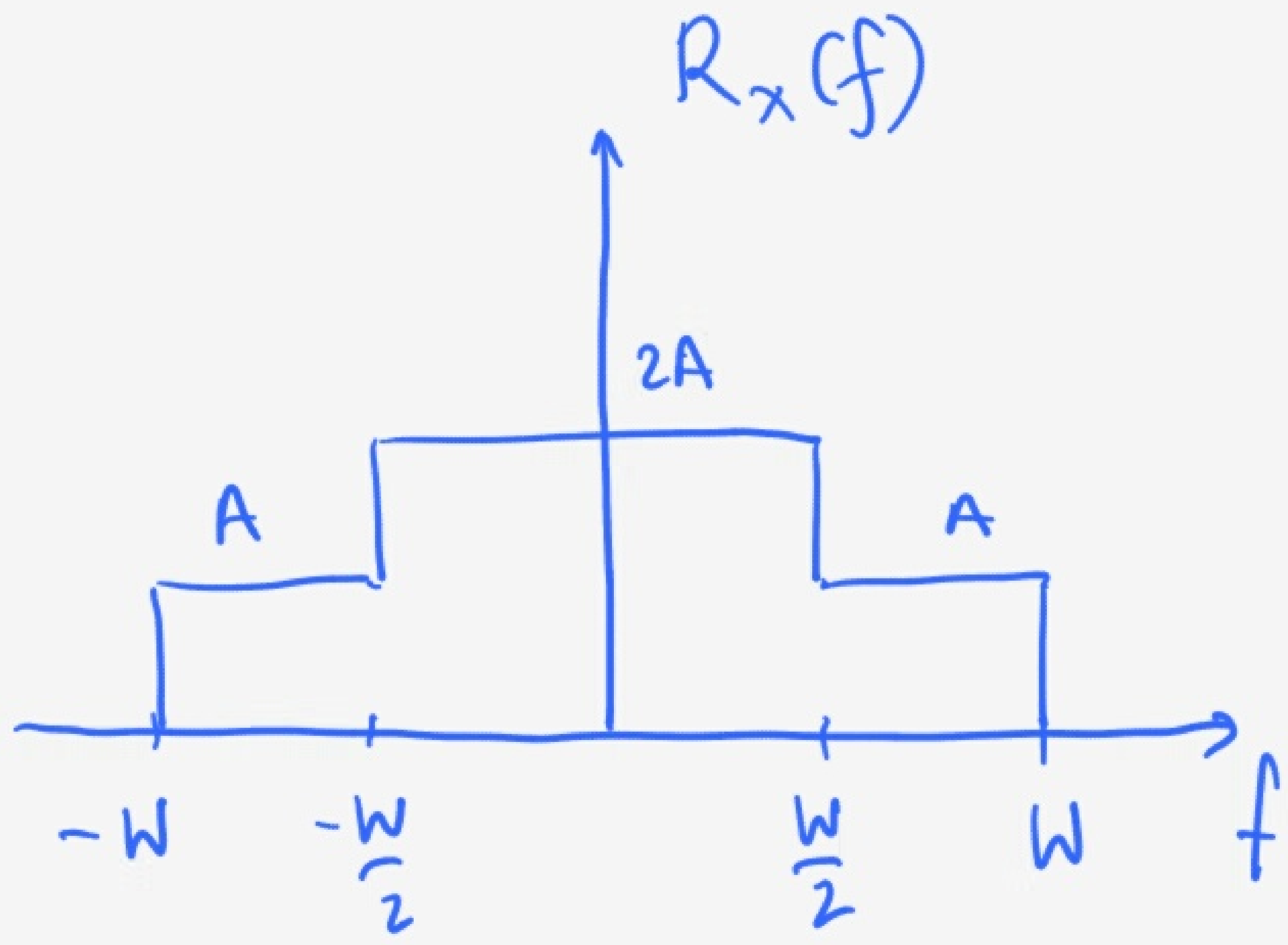
and $9 = c(1-a^2) \Rightarrow 9 = c\left(1-\frac{1}{9}\right)$

$9 = c\left(\frac{8}{9}\right)$ $c = \frac{81}{8}$

Hence we have that the DFT^{-1} of $R(\nu)$ is

$$r_\nu(n) = \frac{c}{b}\left(1/3\right)^{|n|} = \frac{81/8}{9}\left(3^{-1}\right)^{|n|} = \frac{9}{8}3^{-|n|}$$

4.11 $x(t)$ has the power spectral density:



Determine its ACF $r_x(\tau)$

$$R_x(f) = A\Pi\left(\frac{f}{W}\right) + A\Pi\left(\frac{f}{2W}\right)$$

Then we have that:

$$x(ct)\ (c \neq 0) \xrightarrow{\mathcal{F}} \frac{1}{|c|}X\left(\frac{f}{c}\right)$$

$$\mathrm{sinc}(t) \xrightarrow{\mathcal{F}} \Pi(f)$$

$$r_x(\tau) = AW\,\mathrm{sinc}(Wt) + A2W\,\mathrm{sinc}(2Wt)$$