5.1) $X(t)$ has $R_X(f) = \begin{cases} R_0 & |f| \leq B \\ 0 & |f| > B \end{cases}$

$$Y(t) = \frac{dX(t)}{dt}$$

a) Power spectral density of $Y(t)$

$R_Y(f) = |H(f)|^2 R_X(f) \rightarrow Y(f) = H(f)\,X(f)$ $= \mathcal{F}\left\{\left(\frac{\partial}{\partial t}\right)^n x(t)\right\} \xrightarrow{\mathcal{F}}$

$(j2\pi f)^n X(f) \}$ $= Y(f) = (j2\pi f)\,X(f) \Rightarrow |H(f)|^2 = 4\pi^2 f^2$

$R_Y(f) = 4\pi^2 f^2 R_X(f)$

$$R_Y(f) = \begin{cases} 4\pi^2 f^2 R_0 & |f| \leq B \\ 0 & f > B \end{cases}$$

b) Ratio between output power $Y(t)$ $|f| \leq B/2$ and the total output power

$$\int_{-B/2}^{B/2} 4\pi^2 f^2 R_0 \, df = 4\pi^2 R_0 \left(\frac{f^3}{3}\right)_{-B/2}^{B/2} = 4\pi^2 R_0 \left(\frac{(B/2)^3}{3} + \frac{(B/2)^3}{3}\right)$$

$$= \cancel{8}\pi^2 R_0 \left(\frac{B^3}{\cancel{8} \cdot 3}\right) = \pi^2 R_0 \cdot \frac{B^3}{3}$$

Total power:

$$\int_{-B}^{B} 4\pi^2 f^2 R_0 \, df = 4\pi^2 R_0 \left(\frac{f^3}{3}\right)_{-B}^{B} = 4\pi^2 R_0 \left(2\,\frac{(B)^3}{3}\right) =$$

$$= 8\pi^2 R_0 \frac{B^3}{3}$$

$$\text{Ratio} = \frac{\pi^2 R_0 \cdot \frac{B^3}{3}}{8\pi^2 R_0 \cdot \frac{B^3}{3}} = \frac{1}{8}$$

5.16

$X(t) = S_1(t) + N_1(t)$ connected to the input of a lowpass filter

$S_1(t) = A\cos(\omega_0 t + \Phi)$ A and Φ are independent R.V.

$N_1(t)$: white noise with spectral density R_0

$P_A = E\{A^2\}$

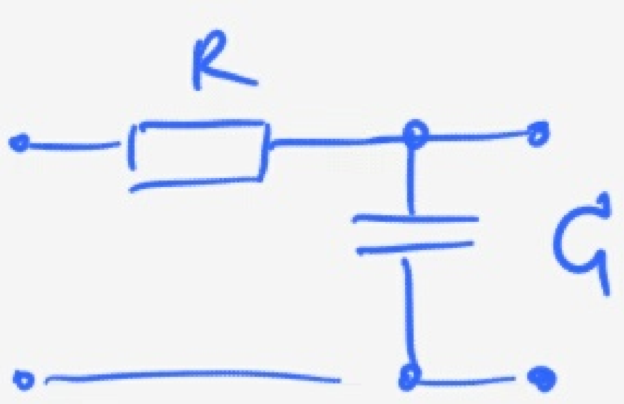
Φ uniform $[0, 2\pi)$

$Y(t) = S_2(t) + N_2(t)$ $\qquad SNR = \dfrac{E\{S_2^2(t)\}}{E\{N_2^2(t)\}}$

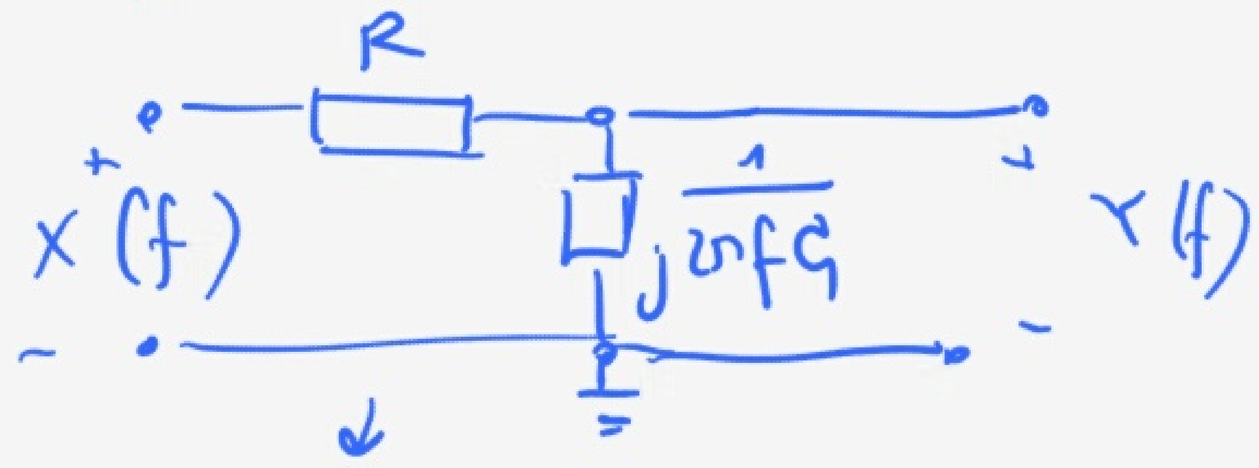
determine $T = RC$ that maximizes SNR.

C and R are linear elements (ideally) hence:

$Y(t) = h(t) * X(t)$ and $Y(f) = H(f)\,X(f)$



Fourier (no transient effect) $\rightsquigarrow$ Transformed Circuit



$$\frac{X(f) - Y(f)}{R} = \frac{Y(f)}{1/j2\pi fC}$$

$$\big(X(f) - Y(f)\big)\frac{1}{j2\pi fC} = R\,Y(f)$$

$$\frac{X(f)}{j2\pi fC} = Y(f)\left(R + \frac{1}{j2\pi fC}\right) \qquad X(f)\,\frac{\frac{1}{j2\pi fC}}{R + \frac{1}{j2\pi fC}} = Y(f)$$

$$X(f)\,\frac{1}{1 + j2\pi fC\cdot R} = Y(f) \qquad H(f) = \frac{1}{1 + j2\pi fT}$$

Two options to calculate power $\nearrow$ $r_y(\tau)$ for $\tau = 0$ convolution 2 exponentials!

$\searrow$ $\int R_y(f)\,df$ ✓

$$R_y(f) = R_x(f)\,|H(f)|^2 \qquad |H(f)|^2 = \frac{1}{1 + (2\pi fRC)^2}$$

$$r_x(\tau) = E\big[\big(A\cos(\omega_0(t+\tau) + \Phi) + N_1(t+\tau)\big)\big(A\cos(\omega_0 t + \Phi) + N_1(t)\big)\big] =$$

$= \{$indep A and $\Phi\} =$

$$+ E[N_1(t+\tau)\,\cancel{A\cos(\omega_0 t+\phi)}]^{\,\emptyset} + E[N_1(t)\,\cancel{A\cos(\omega_0(t+\tau)+\phi)}]^{\,\emptyset}$$

N_1 indp. from rest and

$\emptyset$ average

tutorial 4

$$+ E[N_1(t+\tau)N_1(t)] = \frac{E[A^2]}{2}\cos(\omega_0\tau) + \underbrace{R_0\delta(\tau)}_{\text{white noise}}$$

Taking the fourier transform we will get $R_x(f)$:

$$R_x(f) = \underbrace{\frac{P_A}{4}\left(\delta(f-f_0)+\delta(f+f_0)\right)}_{\text{signal of interest}} + \underbrace{R_0}_{\text{noise}} \quad ; \text{ where } f_0 = \frac{\omega_0}{2\pi}$$

$$R_y(f) = R_x(f)\,|H(f)|^2$$

$$R_y(f) = \frac{P_A}{4}\left(\delta(f-f_0)+\delta(f+f_0)\right)\left(\frac{1}{1+(2\pi fRC)^2}\right) + \frac{R_0}{1+(2\pi fRC)^2}$$

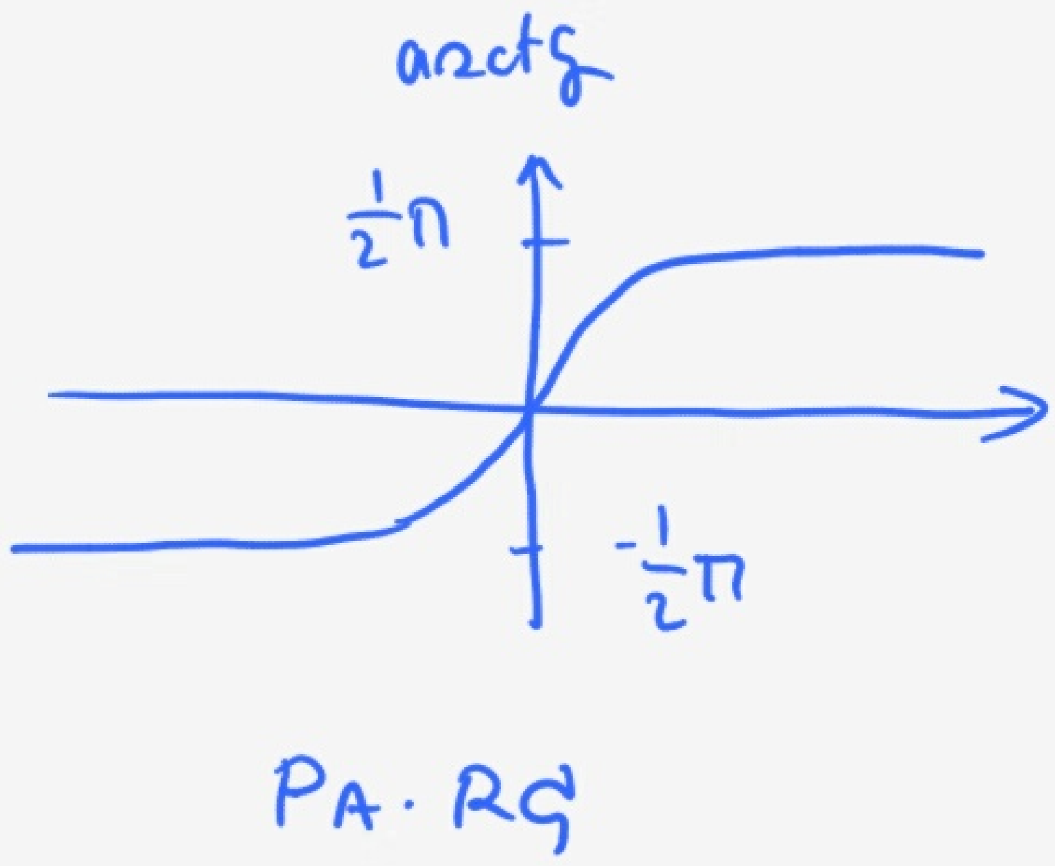
$$P_{S_2} = \int \frac{P_A}{4}\left(\delta(f-f_0)+\delta(f+f_0)\right)\frac{1}{1+(2\pi fRC)^2}\,df =$$

$$= \frac{P_A}{4}\left(\frac{1}{1+(2\pi f_0RC)^2}+\frac{1}{1+(2\pi f_0RC)^2}\right) = \frac{P_A}{2}\,\frac{1}{1+(2\pi f_0RC)^2}$$

$$P_{N_2} = \int_{-\infty}^{\infty}\frac{R_0}{1+(2\pi fRC)^2}\,df = \frac{R_0}{(2\pi RC)^2}\int_{-\infty}^{\infty}\frac{1}{\frac{1}{(2\pi RC)^2}+f^2}\,df = \left\{\int\frac{1}{a^2+x^2}\,dx = \frac{1}{a}\,\text{arctg}\left(\frac{x}{a}\right)\right\}$$

$$= \frac{R_0}{(2\pi RC)^{\cancel{2}}}\cdot(\cancel{2\pi RC})\cdot\text{arctg}(2\pi RCf)\Big]_{-\infty}^{\infty} =$$

$$= \frac{R_0}{2\pi RC}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{R_0}{2RC}$$



Hence the $\text{SNR} = \dfrac{P_A \big/ 1+(\omega_0RC)^2}{R_0 \big/ RC} = \dfrac{P_A\cdot RC}{R_0\left(1+(\omega_0RC)^2\right)}$

we now want to maximize the SNR, as a function of $T = RC$

$$\max\left\{\frac{P_A\,T}{R_0\left(1+(\omega_0T)^2\right)}\right\} = \max\left\{\frac{T}{1+(\omega_0T)^2}\right\}$$

→ Take first derivative $\dfrac{1\cdot(1+(\omega_0 T)^2) - 2T\omega_0^2\cdot T}{(1+(\omega_0 T)^2)^2} =$

$= \dfrac{1-\omega_0^2 T^2}{(1+(\omega_0 T)^2)^2} = \emptyset$ $1 = \omega_0^2 T^2$ $T^2 = 1/\omega_0^2$

$T = \pm 1/\omega_0$] we do not want negative resistors or capacitiers :

$T = 1/\omega_0 \rightarrow \boxed{RC = 1/\omega_0}$

• For the problem to be stricly correct one would have to check if this is actually a maximum by seing that the second order derivate for $T = 1/\omega_0$ is negative. We will however trust the headline in this case.

6.2. $Y(n) = 0.5\,Y(n-1) + X(n)$

$X(n)$ has $\emptyset$ mean   Mean power of $Y(n)$

ACF $r_X(k) = \delta(k)$ $E[Y(n+1)\,Y(n)]$

$Y(n)$ is an order 1 A-R process.

⇒ $Y(n-1)$ and $X(n)$ are independent since $Y(n-1)$ is "made of" previous samples of X and $X(n)$ is white.

$E[Y(n)^2] = E[(0.5Y(n-1) + X(n))(0.5Y(n-1) + X(n))] =$

$= (0.5)^2 E[Y^2(n-1)] + E[X^2(n)] = (0.5)^2 r_y(0) + r_x(0) =$

$r_y(0) = (0.5)^2 r_y(0) + r_x(0)$

$(1-\frac{1}{4})\, r_y(0) = r_x(0)$ $r_y(0) = \frac{4}{3} r_x(0)$ $\boxed{r_y(0) = 4/3}$

• $E[Y(n+1)Y(n)] = E[(0.5Y(n) + X(n+1))(Y(n))] =$

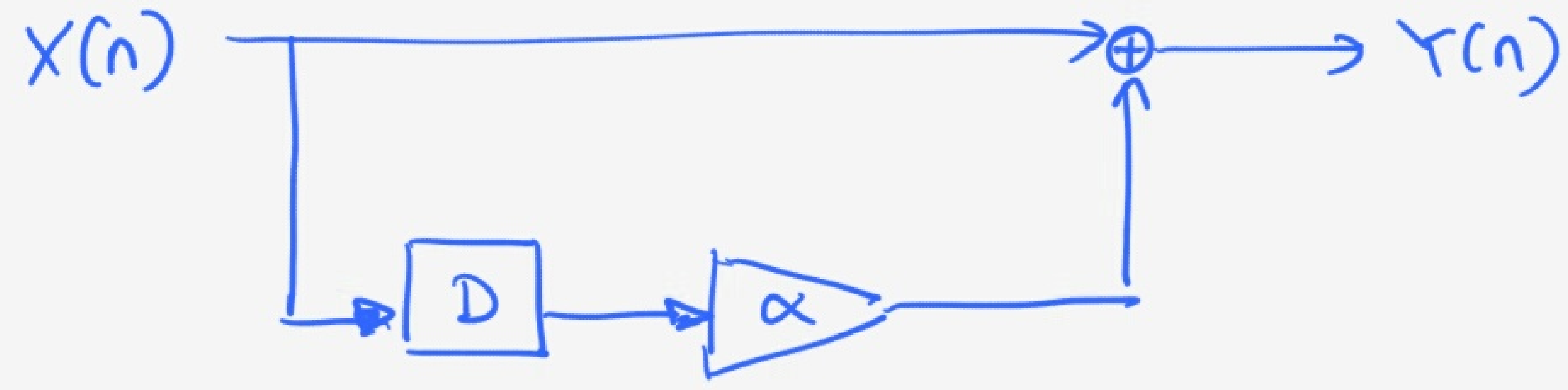
$= E[0.5Y^2(n)] + E[\cancel{X(n+1)Y(n)}]^{\emptyset} =$

indep and $X(n+1)$ has $\emptyset$ mean

$= 0.5 r_x(0)$

$\boxed{r_x(1) = 0.5 r_x(0) = 0.5 \cdot \frac{4}{3} = \frac{2}{3}}$

6.11 WSS $X(n)$ ACF $r_x(k)$



α that achieves the least mean output power as possible.

$Y(n) = \alpha X(n-1) + X(n)$

$E[Y^2(n)] = E[(X(n) + \alpha X(n-1))(X(n) + \alpha X(n-1))] =$

$= r_x(0) + 2\alpha r_x(1) + \alpha^2 r_x(0) = (1+\alpha^2) r_x(0) + 2\alpha r_x(1)$

we want to minimize, we take derivative.

$2\alpha r_x(0) + 2 r_x(1) = 0 \qquad \alpha = \frac{-r_x(1)}{r_x(0)}$

$\boxed{\alpha = \frac{-r_x(1)}{r_x(0)}}$; 2^{nd} order derivative $2 r_x(0) > \emptyset \Rightarrow$ it's a minimum

$\Rightarrow$ Determine ratio $\left.\frac{E[Y(n)^2]}{E[X(n)^2]}\right|_{\alpha = \frac{-r_x(1)}{r_x(0)}}$

$E[Y(n)^2] = \left(1 + \frac{r_x^2(1)}{r_x^2(0)}\right) r_x(0) - 2\frac{r_x(1)}{r_x(0)} r_x(1) =$

$= r_x(0) + \frac{r_x^2(1)}{r_x(0)} - \frac{2 r_x^2(1)}{r_x(0)} = r_x(0) - \frac{r_x^2(1)}{r_x(0)}$

$$\text{Ratio} = \frac{E[Y^2(n)]}{r_x(0)} = \frac{r_x(0) - \frac{r_x^2(1)}{r_x(0)}}{r_x(0)} = 1 - \left(\frac{r_x(1)}{r_x(0)}\right)^2$$