7.1. ACF $r_Y(k)$ for ARMA(1,1) expressed in the parameters σ^2, a_1, b_1.

$$E[X(n)] = \sigma^2$$

$$Y(n) + a_1 Y(n-1) = X(n) + b_1 X(n-1)$$

Can be seen as a linear system

$$Y(\omega) + a_1 e^{-j\omega} Y(\omega) = X(\omega) + b_1 e^{-j\omega} X(\omega) \quad \Big| \; \omega = 2\pi\nu$$

$$Y(\omega)(1 + a_1 e^{-j\omega}) = X(\omega)(1 + b_1 e^{-j\omega})$$

$$H(\omega) = \frac{1 + b_1 e^{-j\omega}}{1 + a_1 e^{-j\omega}}$$

$$|H(\omega)|^2 = \frac{1 + b_1^2 + 2b_1 \cos(2\pi\nu)}{1 + a_1^2 + 2a_1\cos(2\pi\nu)} = \frac{A}{1 + a_1^2 + 2a_1\cos(2\pi\nu)} + B$$

$$B(1 + a_1^2 + 2a_1\cos(2\pi\nu)) + A = 1 + b_1^2 + 2b_1\cos(2\pi\nu)$$

$$B(1 + a_1^2) + A = 1 + b_1^2$$

$$\cancel{2} a_1 B \cancel{\cos(2\pi\nu)} = \cancel{2} b_1 \cancel{\cos(2\pi\nu)} \Rightarrow B = \frac{b_1}{a_1}$$

$$A = 1 + b_1^2 - B(1 + a_1^2) = 1 + b_1^2 - \frac{b_1}{a_1}(1 + a_1^2)$$

$$= \frac{b_1}{a_1} + \frac{1 + b_1^2 - \frac{b_1}{a_1}(1 + a_1^2)}{1 + a_1^2 + 2a_1\cos(2\pi\nu)} = \frac{b_1}{a_1} +$$

$$+ \underbrace{\frac{1 + b_1^2 - \frac{b_1}{a_1}(1 + a_1^2)}{1 - a_1^2}}_{d} \cdot \frac{1 - a_1^2}{1 + a_1^2 + 2a_1\cos(2\pi\nu)} =$$

$$= \frac{b_1}{a_1} + d \cdot \frac{1 - a_1^2}{1 + a_1^2 + 2a_1\cos(2\pi\nu)}$$

$$C = \sigma_X^2 \cdot d$$

$$R_Y(\nu) = |H(\nu)|^2 \underbrace{R_X(\nu)}_{\sigma_X^2} = R_Y(\nu) = \sigma_X^2 \frac{b_1}{a_1} + C \, \frac{1 - a_1^2}{1 + a_1^2 + 2a_1\cos(2\pi\nu)}$$

$$r_Y(k) = \sigma_X^2 \delta(k) \frac{b_1}{a_1} + C(-a_1)^{|k|}$$

7.5. $X(n)$ sinusoidal $S(n)$ and noise $V(n)$ $X(n) = S(n) + V(n)$

ACF for $X(n)$ is

$$r_x(k) = \sigma_s^2 \cos(2\pi\nu_s k) + \sigma_v^2 \delta(k)$$

a_1, a_2 for AR(2) adapted to $X(n)$

$$Y(n) = -a_1 Y(n-1) - a_2 Y(n-2) + Z(n)$$

$$Y^2(n) = -a_1 Y(n-1) Y(n) - a_2 Y(n-2) Y(n) + Z(n) Y(n)$$

$$E[Y^2(n)] = -a_1 E[Y(n-1)Y(n)] - a_2 E[Y(n-2)Y(n)] + E[Z(n)Y(n)]$$

$$r_y(0) = -a_1 r_y(1) - a_2 r_y(2) + E[Z(n)(-a_1 \underbrace{Y(n-1)}_{} - a_2 \underbrace{Y(n-2)}_{} + Z(n))]$$

independent of !

$$\boxed{r_y(0) = -a_1 r_y(1) - a_2 r_y(2) + \sigma_N^2}$$

$$Y(n)Y(n-1) = -a_1 Y^2(n-1) - a_2 Y(n-1)Y(n-2) + Z(n)Y(n-1)$$

$$E[Y(n)Y(n-1)] = -a_1 E[Y^2(n-1)] - a_2 E[Y(n-1)Y(n-2)] + \underbrace{E[Z(n)Y(n-1)]}_{}$$

$$\boxed{r_y(1) = -a_1 r_y(0) - a_2 r_y(1)}$$

indep and $X(n)$ $\emptyset$ mean

$$Y(n)Y(n-2) = -a_1 Y(n-1)Y(n-2) - a_2 Y^2(n-2) + X(n)Y(n-2)$$

$$E[Y(n)Y(n-2)] = -a_1 E[Y(n-1)Y(n-2)] - a_2 E[Y^2(n-2)] + E[\underbrace{Z(n)Y(n-2)}_{}]$$

$$\boxed{r_y(2) = -a_1 r_y(1) - a_2 r_y(0)}$$

indep and $X(n)$ is $\emptyset$ mean

We have the equations

$$\begin{pmatrix} r_y(0) & r_y(1) & r_y(2) \\ r_y(1) & r_y(0) & r_y(1) \\ r_y(2) & r_y(1) & r_y(0) \end{pmatrix} \begin{pmatrix} 1 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sigma^2 \\ 0 \\ 0 \end{pmatrix} \qquad a_1?\ a_2?, \sigma^2?$$

The process $X(n)$ has ACF: $r_x(k) = \sigma_s^2 \cos(2\pi\nu_s k) + \sigma_v^2 \delta(k)$

$$\left.\begin{aligned} r_x(0) &= \sigma_s^2 \cos(\emptyset) + \sigma_v^2 = \sigma_s^2 + \sigma_v^2 \\ r_x(1) &= \sigma_s^2 \cos(2\pi\nu_s) \\ r_x(2) &= \sigma_s^2 \cos(4\pi\nu_s) \end{aligned}\right\}$$

plug in the matrix and solve the system, numerical solution in the solutions

$\frac{\sigma_v}{\sigma_s} \to 0$ $\quad a_1 = -2\cos(2\pi\nu_s)$ } solution and finding limits at the end

$a_2 = 1$

· What happens to the poles of the AR model?

$$Y(n) = -a_1 Y(n-1) - a_2 Y(n-2) + Z(n)$$

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) + Z(n)$$

$$Y(z)(1 + a_1 z^{-1} + a_2 z^{-2}) = Z(n)$$

$$H(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{z^2}{z^2 + a_1 z + a_2}$$

poles $z^2 + a_1 z + a_2 = 0$ $\quad z = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$

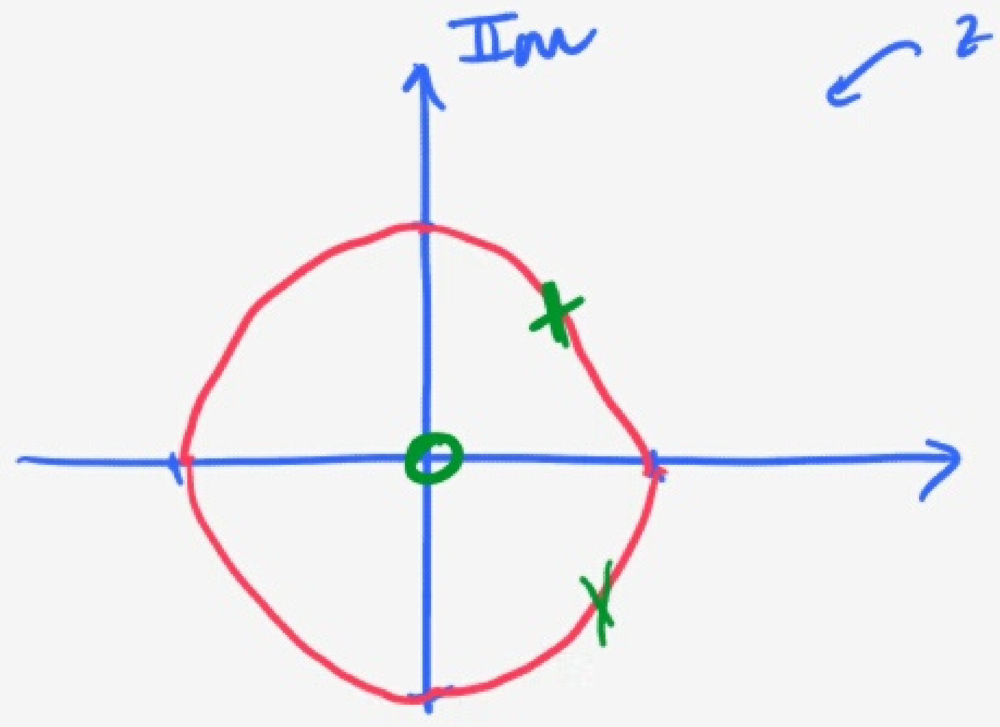
$\frac{\sigma_v}{\sigma_s} \to 0$ $\quad a_1 = -2\cos(2\pi\nu_s)$

$a_2 = 1$

$$z = \frac{2\cos(2\pi\nu_s) \pm \sqrt{4\cos^2(2\pi\nu_s) - 4}}{2} =$$

$$z = \cos(2\pi\nu_s) \pm \sqrt{-(1 - \cos^2(2\pi\nu))} =$$

$$= \cos(2\pi\nu_s) \pm (\sqrt{-1})\sin(2\pi\nu_s) = \cos(2\pi\nu_s) \pm j\sin(2\pi\nu_s)$$



z-plane

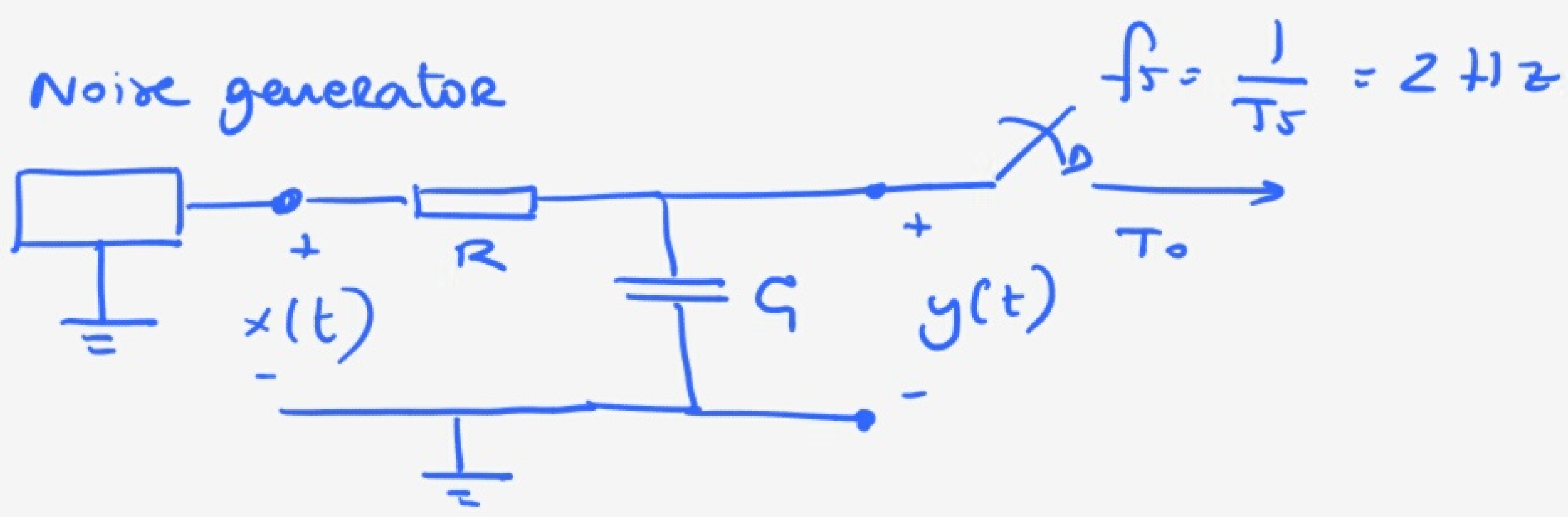
unitary circle

o zeros (double in this case)

x poles

· a pole is a point where $H(z)$ goes to infinity! Hence we will only have a system that outputs a sinusoid of frequency ν_s! ⇒ we have an oscillator at frequency ν_s.

7.7.



$$Y(n) + aY(n-1) = X(n) \qquad E[X^2(n)] = \sigma^2$$

$$\left.\frac{dy(t)}{dt}\right|_{t=kT_S} \approx \frac{y(kT_S) - y((k-1)T_S)}{T_S}$$ (1st order aproximation of derivative)

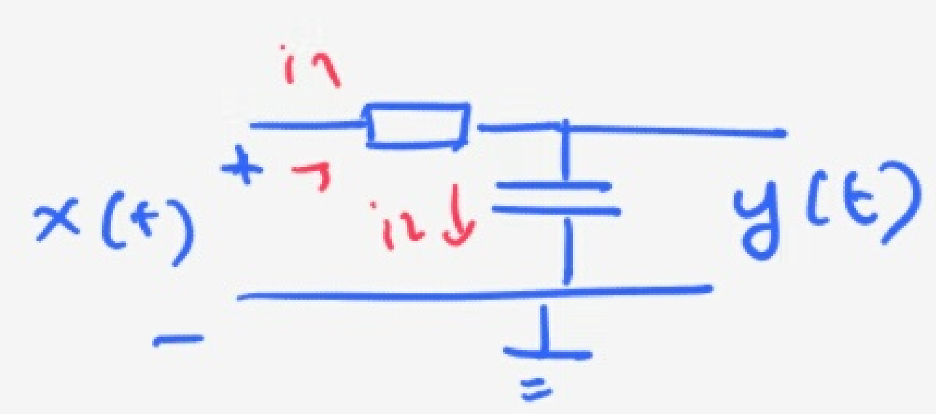
$$\hat{a} = \left[\frac{1}{N}\sum_{n=0}^{N-1} y(n-1)^2\right]^{-1} \frac{1}{N}\sum_{n=0}^{N-1} y(n)y(n-1) \approx -0.5$$

- $\omega_c = 1/RC_1$?

$$Y(n) - 0.5\,Y(n-1) = X(n)$$

We try to identify the coefficients of the RC filter with the AR model

An RC circuit



$$i_{C_1}(t) = C_1 \frac{dV_c(t)}{dt}$$

Current over a capacitor : $i_2(t) = C_1 \dfrac{dy(t)}{dt}$

$$i_1(t) = i_2(t) \qquad \frac{x(t) - y(t)}{R} = C_1 \frac{dy(t)}{dt}$$

$$\left.\frac{dy(t)}{dt}\right|_{t=kT_S} \approx \frac{y(kT_S) - y((k-1))T_S}{T_S}$$

$$\frac{x(k) - y(k)}{R} = \frac{C_1}{T_S}\left(y(k) - y(k-1)\right), \qquad x(k) - y(k) = \frac{RC_1}{T_S}\left(y(k) - y(k-1)\right)$$

$$x(k) = y(k)\left(1 + \frac{RC}{T_S}\right) - \frac{RC}{T_S}\,y(k-1)$$

$$y(n) - \frac{\frac{RC}{T_s}}{\left(1+\frac{RC}{T_s}\right)} y(n-1) = \frac{x(n)}{\left(1+\frac{RC}{T_s}\right)} ;$$

$$y(n) - \frac{RC}{T_s + RC} y(n-1) = \frac{T_s\, x(n)}{T_s + RC}$$

$$a = \frac{-RC}{T_s + RC} = \left\{ T_s = \frac{1}{2} \right\} = \frac{-RC}{\frac{1}{2} + RC} = -0.5$$

$$\frac{RC}{0.5 + RC} = 0.5 \qquad RC = 0.25 + 0.5RC$$

$$0.5RC = 0.25 \Rightarrow RC = \frac{1}{2}$$

$$a_1 = -\frac{\sigma_s^2 \cos(2\pi\nu_s)\left(\sigma_s^2 + \sigma_v^2 - \sigma_s^2 \cos(4\pi\nu_s)\right)}{(\sigma_s^2 + \sigma_v^2)^2 - \sigma_s^4 \cos^2(2\pi\nu_s)}$$

divide numerator and denominator by σ_s^4

$$a_1 = -\frac{\sigma_s^2/\sigma_s^2 \cos(2\pi\nu_s)\left(\sigma_s^2/\sigma_s^2 + \sigma_v^2/\sigma_s^2 - \sigma_s^2/\sigma_s^2 \cos(4\pi\nu_s)\right)}{(\sigma_s^2/\sigma_s^2 + \sigma_v^2/\sigma_s^2)^2 - \frac{\sigma_s^4}{\sigma_s^4} \cos^2(2\pi\nu_s)}$$

$$a_1 = \frac{-\cos(2\pi\nu_s)\left(1 + \frac{\sigma_v^2}{\sigma_s^2} - \cos(4\pi\nu_s)\right)}{(1 + \sigma_v^2/\sigma_s^2)^2 - \cos^2(2\pi\nu_s)}$$

$\cos(2x) = \cos^2(x) - \sin^2(x)$

$$\lim_{\frac{\sigma_v}{\sigma_s} \to 0} a_1 = \frac{-\cos(2\pi\nu_s)\left(1 - \cos(4\pi\nu_s)\right)}{1 - \cos^2(2\pi\nu_s)} =$$

$\sin^2(x) = 1 - \cos^2(x)$

$$= \frac{-\cos(2\pi\nu_s)\left(1 - \cos^2(2\pi\nu_s) + \sin^2(2\pi\nu_s)\right)}{1 - \cos^2(2\pi\nu_s)} = \frac{-\cos(2\pi\nu_s)\left(1 - \cos^2(2\pi\nu_s)\right) 2}{\left(1 - \cos^2(2\pi\nu_s)\right)}$$

$$= -2\cos(2\pi\nu_s)$$

$$a_2 = -\frac{(\sigma_s^2 + \sigma_v^2)\sigma_s^2 \cos(4\pi\nu_s) - \sigma_s^4 \cos^2(2\pi\nu_s)}{(\sigma_s^2 + \sigma_v^2)^2 - \sigma_s^4 \cos^2(2\pi\nu_s)} =$$

$$= -\frac{\left(\sigma_s^2/\sigma_s^2 + \sigma_v^2/\sigma_s^2\right)\frac{\sigma_s^2}{\sigma_s^2} \cos(4\pi\nu_s) - \frac{\sigma_s^4}{\sigma_s^4} \cos^2(2\pi\nu_s)}{\left(\sigma_s^2/\sigma_s^2 + \sigma_v^2/\sigma_s^2\right)^2 - \sigma_s^4/\sigma_s^4 \cos^2(2\pi\nu_s)} = -\frac{\left(1 + \frac{\sigma_v^2}{\sigma_s^2}\right)\cos(4\pi\nu_s) - \cos^2(2\pi\nu_s)}{\left(1 + \frac{\sigma_v^2}{\sigma_s^2}\right)^2 - \cos^2(2\pi\nu_s)}$$

$$\lim_{\frac{\sigma_v}{\sigma_s} \to 0} a_2 = \frac{(1)\cos(4\pi\nu_s) - \cos^2(2\pi\nu_s)}{1 - \cos^2(2\pi\nu_s)} = \left(\cos(2x) = 2\cos^2(x) - 1 \right)$$

$$= \frac{\cos^2(2\pi\nu_s) - 2\cos^2(2\pi\nu_s) + 1}{1 - \cos^2(2\pi\nu_s)} = \boxed{1}$$

Then, $\sigma_z^2 = r_y(0) + r_y(1)a_1 + r_y(2)a_2$

$r_y(0) = \sigma_s^2 + \sigma_v^2$

$r_y(1) = \sigma_s^2 \cos(2\pi\nu_s)$

$r_y(2) = \sigma_s^2 \cos(4\pi\nu_s)$

$$\sigma_z^2 = (\sigma_s^2 + \sigma_v^2) + \sigma_s^2 \cos(2\pi\nu_s)(-2\cos(2\pi\nu_s)) + \sigma_s^2 \cos(4\pi\nu_s)$$

$$\sigma_z^2 = (\sigma_s^2 + \sigma_v^2) - 2\sigma_s^2 \cos(2\pi\nu_s)^2 + \sigma_s^2(2\cos(2\pi\nu_s) - 1)$$

$$\sigma_z^2 = (\sigma_s^2 + \sigma_v^2) - \sigma_s^2 = \sigma_v^2$$

Note now that if $\sigma_v / \sigma_s \to 0$ and the process $s(n)$ has a variance, σ_s is bounded $\Rightarrow \sigma_v = 0$ $\quad \underline{\sigma_z \to 0}$