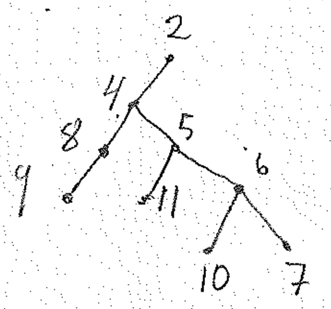
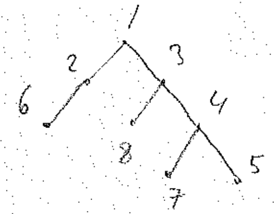
Trees:

Increasing binary tree:
- Unique root
- Each vertex is a unique natural number
- Each vertex is smaller than its children (successors)
- Each vertex has (exactly) either
  - a left child,
  - a right child,
  - a left and a right child, or
  - no children.

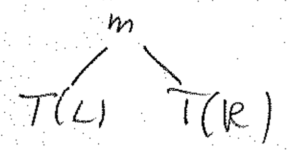




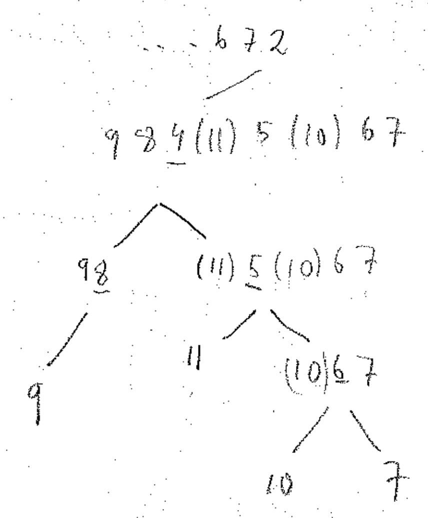
Let $w = w_1 w_2 \dots w_n$ be a word on $\mathbb{P} = \{1, 2, 3, \dots\}$ with no repeated letters. Define an increasing binary tree as follows:

- $T(\emptyset) = \emptyset$
- If m is the smallest letter in w, write $w = LmR$ (concatenation). Then

$T(w) =$

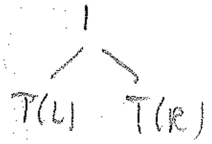


9 8 4 (11) 5 (10) 6 7 2 $\rightarrow$



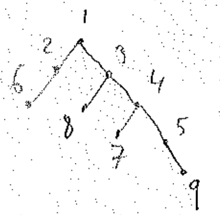
<u>Fact</u>: The correspondance $w \to T(w)$ is a bijection between $\mathfrak{S}_n$ and increasing binary trees on $\{1,\dots,n\}$

Easy by induction $L1R \to$



<u>Def</u>:
Let $w = w_1 w_2 \dots w_n \in \mathfrak{S}_n$, and set $w_0 = w_{n+1} = 0$.
$w_i \in [n]$ is a

double rise (double ascent) if $w_{i-1} < w_i < w_{i+1}$
-11- fall -11- descent if $w_{i-1} > w_i > w_{i+1}$
peak if $w_{i-1} < w_i > w_{i+1}$
valley if $w_{i-1} > w_i < w_{i+1}$



$\leftrightarrow$

0	6	2	1	8	3	7	4	5	9	0
	p	df	v	p	v	p	v	dr	p	

Element w_i of w	Vertex w_i has precisely the following successors
double rise	right $L = \emptyset$
double fall	left $R = \emptyset$
valley	left and right
peak	none

<u>Def</u>: A permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ is <u>alternating</u> if $\pi_1 > \pi_2 < \pi_3 > \dots$, and <u>reverse alternating</u> if $\pi_1 < \pi_2 > \pi_3 < \dots$. Let E_n be the number of alternating permutations in $\mathfrak{S}_n$.

Prop 1.5.3

(a) # increasing binary trees on $[n]$ is $n!$

(b) # such trees for which exactly k vertices have left successors is $A(n, k+1)$

(c). The number of complete increasing binary trees on $[2n+1]$ is E_{2n+1}. (Each vertex is either an endpoint or has two successors).

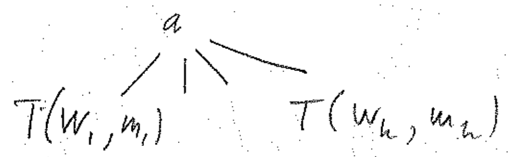
Proof: (b) $w_{i-1} > w_i \Leftrightarrow w_i$ double fall or valley

(c). Alternating $\Leftrightarrow$ All w_i valleys or peaks.

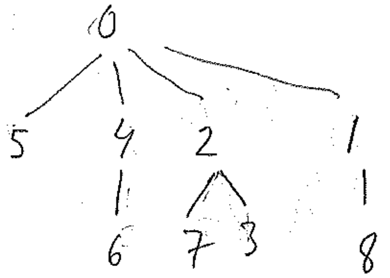
Unordered increasing ordered tree on $\mathbb{P}$.

- Each vertex is smaller than its successors.
- No order among the successors (no left/right)

If $w = w_1 w_2 \ldots w_n$ is a word with no repetitions and $a < w_i$ for all i, let $T'(\emptyset, a) = a$

$T'(w, a) =$ 

$w = m_1 w_1 m_2 w_2 \ldots m_h w_h$ $\quad$ $m_1 < \ldots < m_h$ are the left-to-right minima

$T'(\underline{5}\,\underline{4}\,6\,\underline{2}\,7\,3\,\underline{1}\,8, 0) =$ 

The correspondence $w \to T'(w)$ is a bijection between $\mathfrak{S}_n$ and increasing trees on $[0,n]$.

Prop 1.5.5

(a). # incr. trees on $[n+1]$ is $n!$

(b). # such trees with k successors of the root is $c(n,k)$.

(c). # such trees with k endpoints is $A(n,k)$.

Prop 1.6.1

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec(x) + \tan(x) = \frac{1}{\cos x} + \frac{\sin(x)}{\cos(x)}$$

Proof: Let $A_n = \{\text{alternating}\} \subseteq \mathfrak{S}_n$
$R_n = \{\text{rev-alternating}\} \subseteq \mathfrak{S}_n$

A permutation in $A_{n+1} \cup R_{n+1}$ may be uniquely written as $w = L\,1\,R$ where $R = w_{k+2} w_{k+3} \cdots w_{n+1}$ is alternating and $L = w_1 w_2 \cdots w_k$, and $w_k w_{k-1} \cdots w_1$ is alternating

Since $|A_{n+1}| = |R_{n+1}|$, we have

$$2 E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k} \quad , n \geq 1$$

Let $f(x) = \sum_n E_n \frac{x^n}{n!}$

$$2 f'(x) = \sum_{n \geq 0} 2 E_{n+1} \frac{x^n}{n!} = 2 + \sum_{n\geq 1} \left(\sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k} \right) \frac{x^n}{n!}$$

$$= 1 + f^2 \quad , \quad f(0) = 0$$

This has a unique solution $\sec(x) + \tan(x)$ $\square$

Permutations of multisets

$w = w_1 \dots w_n \in \mathfrak{S}_M$

Inversions: $\quad \mathrm{inv}(w) = \#\{(i,j) : w_i > w_j ,\ i<j\}$

q-multinomial number:

$$\binom{n}{a_1,\dots,a_m}_q = \frac{n!_q}{a_1!_q \cdots a_m!_q} \qquad n = \sum a_i$$

Prop: Let $M = \{1^{a_1},\dots, m^{a_m}\}$. Then

$$\sum_{w\in\mathfrak{S}_M} q^{\mathrm{inv}(w)} = \binom{n}{a_1,\dots,a_m}_q$$

Proof: Partition $[n]$ according to $a_1,\dots,a_m$:

$[n] = A_1 \cup A_2 \cup \dots \cup A_m$, where

$|A_i| = a_i$ and if $x\in A_i$, $y \in A_j$, $i<j$, then $x<y$.

Define $\varphi : \mathfrak{S}_M \times \mathfrak{S}(A_1)\times\dots\times\mathfrak{S}(A_m) \to \mathfrak{S}_n$ by

$(w, \pi^1, \dots, \pi^m) \longrightarrow \pi$:

- The i's in w are replaced by A_i and ordered as π^i.

Bijection with $\mathrm{inv}(\pi) = \mathrm{inv}(w) + \mathrm{inv}(\pi^1) + \dots + \mathrm{inv}(\pi^m)$

$\therefore$ $$n!_q = \sum_{\pi\in\mathfrak{S}_n} q^{\mathrm{inv}(\pi)} = \sum_{w}\sum_{\pi^1}\dots\sum_{\pi^m} q^{\mathrm{inv}(w)}\cdot q^{\mathrm{inv}(\pi^1)} \cdots q^{\mathrm{inv}(\pi^m)}$$

$$= \Big(\sum_{w\in\mathfrak{S}_M} q^{\mathrm{inv}(w)}\Big)\, a_1!_q\, a_2!_q \cdots a_m!_q$$

$\square$

Let $q = p^r$ be a prime-power and F_q a field with q elements.

Let $V_n(q)$ be an n-dimensional vectorspace over F_q ($V_n(q) \cong F_q^n$).

<u>Prop:</u> The number $G(n,k)$ of k-dimensional subspaces in $V_n(q)$ is $\binom{n}{k}_q$.

<u>Proof</u>: Note that the number a vectors in a j-dimensional subspace is q^j, since we may chose a basis $b_1, \ldots, b_j$ and unique coefficients $\lambda_1, \ldots, \lambda_j \in F_q$.

Let $N(n,k) = \#\{(v_1, \ldots, v_k) : v_1, \ldots, v_k \in V_n(q) \text{ lin. ind.}\}$

$$= (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{k-1})$$

any nonzero vector

take any vector in $V_n(q) \setminus \langle v_1 \rangle$

$V_n(q) \setminus \langle v_1, v_2 \rangle$

$$N(n,k) = G(n,k)(q^k - 1) \cdots (q^k - q^{k-1})$$

chose the subspace $\langle v_1, \ldots, v_k \rangle$ first.

□