$\boxed{F4}$ Now we have defined the Lebesgue measure.

Having the measure we can start to integrate.

The idea is as follows, if $-N\varepsilon < f(x) < N\varepsilon$

$$\int_a^b f(x)\,dx \geqslant \sum_{j=-N}^{N} j\varepsilon \; m\big(\underbrace{\{x\,;\, j\varepsilon \leq f(x) < (j+1)\varepsilon\}}_{A_j}\big) \qquad (1)$$

$\Rightarrow$ must be measurable!

and $$\int_a^b f(x)\,dx \leq \sum_{j=-N}^{N} (j+1)\varepsilon \; m\big(\{x\,;\, j\varepsilon \leq f(x) < (j+1)\varepsilon\}\big)$$

so we may approximate $\int_a^b f(x)\,dx$ from above and below with an error of $(b-a)\varepsilon$ at most.

~~Rewriting this~~ Notice that the left hand side is the natural definition of the integral of $\sum_{j=-N}^{N} j\varepsilon\, \chi_{A_j}(x)$

<u>Definition</u>: Let $e(x)$ be a function with finitely many values, ~~and~~

$$e(x) = \sum_{j=1}^{N} a_j \chi_{A_j}(x) \qquad A_j \subset [a,b] \quad A_j \text{ measurable}$$

then the integral of $e(x)$ from a to b is defined to be

$$\int_a^b e(x)\,dx = \sum_{j=1}^{N} a_j\, m(A_j).$$

For a general function $f(x)$ we define

$$\int_a^b f(x)\,dx = \sup_{e \leq f} \int_a^b e(x)\,dx.$$

Notice that the value of the integral of ~~A~~ f are only certain to be right when A_j are measurable. That is when $\underbrace{\{x \,;\, j\varepsilon \le f(x) \le (j+1)\varepsilon\}}$ are measurable.

But ~~the~~ it is enough to assume that

$\{x \,;\, a < f(x)\}$ is measurable for all $a \in \mathbb{R}$

(since $A_j = \left(\bigcup_{k=1}^{\infty} \{x \,;\, j\varepsilon - \frac{1}{k} < f(x)\} \right) \setminus \{x \,;\, (j+1)\varepsilon < f(x)\}$.

~~The question is: what is are the class of~~

Definition: We say that $f(x)$ is measurable on D if the set $\{x \in D \,;\, a < f(x)\}$ is measurable for all $a \in \mathbb{R}$

Proposition: ~~All~~ f continuous on $D \Rightarrow f$ measurable.

Proof: f continuous $\Rightarrow f^{-1}(\underbrace{(a,\infty)}_{\text{open}})$ is open

open sets are measurable $\Rightarrow f$ is measurable.

Proposition: f & g measurable $\Rightarrow kf$ $(k \in \mathbb{R})$, $f+g$, $f \cdot g$, $\sup(f,g)$ and $|f|$ are measurable.

Proof: (simple...) For $\sup(f,g) = h$ we get

$$h^{-1}((a,\infty)) = f^{-1}(a,\infty) \cup g^{-1}(a,\infty),$$

writing $fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$ etc...

Proposition: Let f_i be a sequence of measurable functions

then $\sup f_i$, $\inf f_i$, $\limsup_{j\to\infty} f_j(x)$, $\liminf_{j\to\infty} f_j(x)$

are measurable. $\lim_{j\to\infty} f_j(x)$ measurable (if lim exists)

Proof: Consider, say $\inf f_i = f(x)$ if we want

to show that $f(x)$ is measurable then

we need to show the following set to be

$$\{x;\ f(x) > a\} = \underbrace{\bigcup_{j=1}^{\infty} \{x;\ f_j > a\}}$$

countable union of measurable sets. is measurable.

Similarly if

$\liminf_{j\to\infty} f_j(x) = f(x)$ then

$$f(x) = \lim_{j\to\infty} \inf_{m\geq j} f_j(x)$$

$\cancel{\{f(x) > a\} = \inf_{m\geq 1} \sup}$ Thus

$$\{f(x) > a\} = \underbrace{\bigcap_{j=1}^{\infty} \Big(\underbrace{\bigcup_{m=j}^{\infty} \{f_j(x) > a\}}_{\text{measurable}} \Big)}_{\text{measurable.}}$$

etc. ∎

Notice that these results are only possible to prove because we have ~~[illegible]~~ that countable intersections and unions of measurable sets are measurable.

Thm (Monotone convergence thm): Let $|f_j| \leq M$ be an increasing sequence of measurable functions on measurable D and $-M \leq \lim_{j\to\infty} f_j(x) = f(x) \leq M$. Then

i) $f(x)$ is measurable and integrable.

ii) $\lim_{j\to\infty} \int_D f_j(x)\,dx = \int_D f(x)\,dx$.

$D \to$ Bdd domain.

Example: $f_j(x) = \begin{cases} 1 & \text{if } x = q_k \quad k \leq j \\ 0 & \text{else} \end{cases}$ on $[0;1]$

Then $f = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{else} \end{cases}$

is integrable and

$$\underbrace{\int_0^1 f_j(x)\,dx}_{=0} \to \int_0^1 f(x)\,dx = 0.$$

We have been able to overcome the big problem we discussed in the first lecture.

Proof (Mon. conv. Thm): Since $f_j \nearrow f \Rightarrow f_j \leq f$ for all j

$$\Rightarrow \int_D f_j(x)\,dx \leq \int_D f_j(x)\,dx$$ need the reverse ineq.

~~The good thing is that we may~~ Let $\lambda < 1$ and $E_j = \{x;\ \lambda f(x) \leq f_j(x)\}$ then $\bigcup E_j = D$

(since if $\lambda f(x) > f_j(x)$ for all j then $f_j(x) \not\to f(x) > \lambda f(x)$

Therefore, since $\lim_{j\to\infty} m(E_j) = m(D)$ by continuity of measures

$\exists_j > 0$ s.t. $m(D \setminus E_j) < \varepsilon$ and thus

$$\int_D f(x)\,dx = \int_{D \cap E_j} f(x)\,dx + \underbrace{\int_{D \setminus E_j} f(x)\,dx}_{< M\varepsilon} \leq \frac{1}{\lambda} \int_{D \cap E_j} f_j(x)\,dx + M\varepsilon \leq \frac{1}{\lambda} \int_{D} f_j(x)\,dx + M\varepsilon$$

The only place we use that f is bdd.

Since $\varepsilon > 0$ is arbitrary (tends to 0 as $j \to \infty$, actually) we may pass to the limit $j \to \infty$

$$\int_D f(x)\,dx \leq \frac{1}{\lambda} \lim_{j \to \infty} \int_D f_j(x)\,dx \qquad \text{for all } \lambda < 1$$

send $\lambda \to 1^-$ gives the Theorem. ∎

If we want to generalize the theorem ~~...~~ to unbounded functions we should work with this assumption!

But before we do this we need to ~~...~~ define integrable.

Def: If f is 1) measurable

2) $\int_D |f(x)|\,dx < \infty$

then we say that f is integrable on D.

<u>Lemma</u>: Assume that f is integrable on D then for every $\varepsilon > 0$ $\exists \delta_\varepsilon > 0$ s.t. if $\Sigma \subset D$ satisfies $m(\Sigma) < \delta_\varepsilon$ then

$$\int_\Sigma |f(x)|\,dx < \varepsilon.$$

Proof: No loss of generality to assume $f > 0$.

Then $\int_D f(x)\,dx = \sup_{\varphi} \int_D \varphi(x)\,dx$ · bounded (f integrable)

Thus $\exists$ φ = simple function $\leq f(x)$ s.t.

$$\int_D \underbrace{f(x) - \varphi}_{\geq 0}\,dx < \varepsilon/2 .$$

Since φ is simple it takes only finitely many values $\Rightarrow$ $\sup \varphi = s$ is well defined and finite.

Let $\delta_\varepsilon = \frac{\varepsilon}{2s}$ then, if $m(E) < \delta_\varepsilon$

$$\int_E f(x)\,dx = \underbrace{\int_E \big(f(x) - \varphi(x)\big)\,dx}_{\leq \int_D f - \varphi \, < \varepsilon/2} + \int_E \underbrace{\varphi(x)}_{< s}\,dx < \frac{\varepsilon}{2} + s\, m(E) \leq \varepsilon.$$

∎

Corollary: Let $0 \leq f_i(x) \to f(x)$ ~~then~~ and $f_i(x)$ increasing

$$\lim_{j \to \infty} \int_D f_j(x)\,dx = \int_D f(x)\,dx \qquad (\text{May be } \infty)$$

Proof: As monotone convergence thm using the Lemma where we used $f(x) \leq M$ before.

∎

From basic real analysis we are also interested in interchanging limits and integrals. The next Theorem is useful.

Theorem (Fatou's Lemma).

Let $f_j(x)$ be a sequence of integrable functions on D. Then

$$\int_D \liminf_{j\to\infty} (f_j)\, dx \le \liminf_{j\to\infty} \int_D f_j(x)\,dx$$

<u>Proof</u>: For any $k \geq n$ we have

$$\underbrace{\inf_{j\geq n} f_j(x)}_{\text{measurable.}} \le f_k(x) \qquad \text{thus}$$

$$\int_D \inf_{j\geq n} f_j(x)\,dx \le \int_D f_k(x)\,dx \qquad \text{all } k \geq n$$

$$\Rightarrow \int_D \inf_{j\geq n} f_j(x)\,dx \le \inf_{k\geq n} \int_D f_k(x)\,dx.$$

Notice that $\liminf\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty}\left(\inf\limits_{j\geq n} f_j(x)\right)$

thus taking limits on both sides we get

$$\lim_{n\to\infty}\Big(\int_D \underbrace{\inf_{j\geq n} f_j(x)}_{\text{increasing in } n}\,dx\Big) \le \liminf_{j\to\infty} \int_D f_j(x)\,dx$$

Apply the monotone convergence on the left side gives

$$\int_D \liminf_{j\to\infty} f_j(x)\,dx \le \lim_{n\to\infty}\Big(\int_D \inf_{j\geq n} f_j(x)\,dx\Big) \le \liminf_{j\to\infty} \int_D f_j(x)\,dx$$

change into $=$.

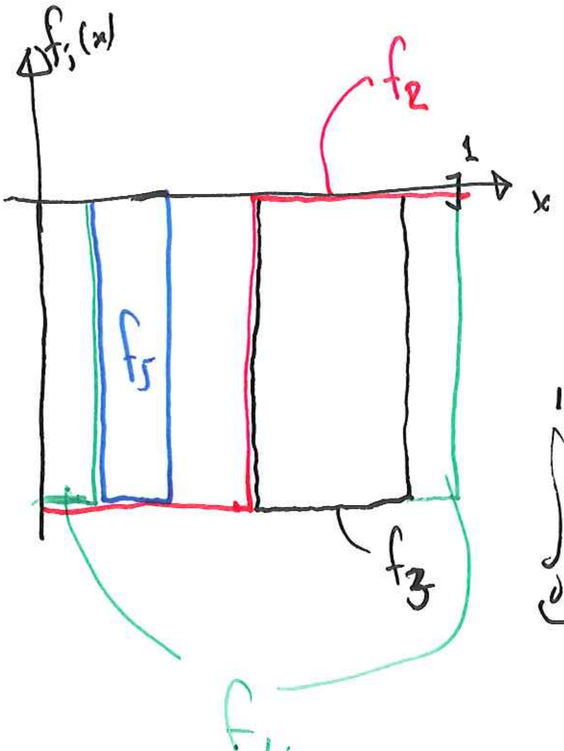
When seeing such a Thm we need to figure out to what extent the Thm is optimal. Why inequality for instance

<u>Example</u>: ~~Let~~ $\cancel{f_1(x) = \{ -1}$

Let $f_2(x) = \begin{cases} -1 & 0 \le x < \frac{1}{2} \\ 0 & \text{else} \end{cases}$, $\quad f_3(x) = \begin{cases} -1 & \frac{1}{2} \le x < \frac{1}{2} + \frac{1}{3} \\ 0 & \text{else} \end{cases}$

$$f_4(x) = \begin{cases} -1 & \frac{1}{2} + \frac{1}{3} \le x \pmod 1 \le \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\ 0 & \text{else} \end{cases} \quad \ldots$$

$$f_j(x) = \begin{cases} -1 & \text{if } \sum_{k=2}^{j-1} \frac{1}{k} \le x \pmod 1 < \sum_{k=2}^{j} \frac{1}{k} \\ 0 & \text{else} \end{cases}$$



Then $\int_0^1 f_j(x)\,dx = \frac{-1}{j}$

and $\liminf_{j\to\infty} f_j(x) = -1$

thus

$$\underbrace{\int_0^1 \liminf_{j\to\infty} f_j(x)\,dx}_{=-1} < \liminf_{j\to\infty} \int_0^1 f_j(x)\,dx = 0.$$

Theorem: (Dominated convergence thm).

Let f_j be a sequence of integrable functions on D such that a) $f_n \to f$ a.e

b) $\exists g$, g integrable, s.t. $|f_j(x)| \leq g(x)$.

Then f is integrable and

$$\lim_{j\to\infty} \int_D f_j(x)\,dx = \int_D f(x)\,dx. \qquad (1)$$

Proof Since $f_j(x) \to f(x)$ $f(x)$ has to be measurable

Since $|f_j| \leq g$ it follows that $|f| \leq g$

and thus f is integrable.

It remains to show (1)

Since $f_j \to f$ it follows that $\liminf\limits_{j\to\infty} f_j(x) = f(x)$

Thus from Fatou's Lemma

$$\liminf_{j\to\infty} \int_D f_j(x)\,dx \geq \int_D f(x)\,dx,$$

and applying the same to $-f_j$ we get

$$\left.\begin{aligned} &\underbrace{\liminf_{j\to\infty} \int_D (-f_j(x))\,dx}_{= -\limsup\limits_{j\to\infty} \int_D f_j(x)\,dx} \geq -\int_D f(x)\,dx \end{aligned}\right\} \Rightarrow \int_D f(x)\,dx \geq \limsup_{j\to\infty} \int_D f_j\,dx$$

Thus

$$\liminf_{j\to\infty} \int_D f_j(x) \geq \int_D f\,dx \geq \limsup_{j\to\infty} \int_D f_j\,dx.$$

$\blacksquare$

Example (why $|f_j| \le g$ in dom conv thm).

Let $f_j(x) = \begin{cases} 4^n & x \in (2^{-n-1}, 2^{-n}) \\ 0 & \text{else.} \end{cases}$

Then $$\int_0^1 f_j(x)\,dx = \int_{2^{-n-1}}^{2^{-n}} 4^n\,dx = 2^{n-1} \longrightarrow \infty$$

and $$\lim_{j \to \infty} f_j(x) = 0 \quad \text{for all } x.$$

Thus the dominated convergence thm does not hold without the assumption $|f_j| \le g$.

Remark: This Theorem is what we have been looking for! Now we know that convergent sequences of functions converge to something that is integrable!

The assumption that $|f_j| \le g$ is not a fault of the definition of the integral. The sequence f_j in the example should have integrals converging to ∞ even though $f_j(x) \to 0$ for each x. We want that - any other definition of the integral where this didn't happen would be wrong!

We end this lecture ~~by~~ with ~~two~~ interesting results about convergence and approximation.

Thm (Egoroff's Thm): Let D be a bounded measurable set and let $f_i(x) \to f(x)$ for each $x \in D$, f_i ~~integrable~~ measurable (thus f ~~integrable~~ measurable).
Then ~~∃E~~ for every $\varepsilon > 0$ $\exists E_\varepsilon \subset D$ s.t.

$f_i(x) \to f(x)$ uniformly on E_ε and

$m(D \setminus E_\varepsilon) < \varepsilon$.

Proof: We define the bad set (where f_i is far from f) according to

$$B_n(k) = \bigcup_{j=n}^{\infty} \{x \in D ; |f_j(x) - f(x)| \geq \tfrac{1}{k}\}$$

Since $f_i(x) \to f(x)$ it follows that

$$\bigcap_{n=1}^{\infty} B_n(k) = \emptyset \quad \Rightarrow \quad m(B_n(k)) \to 0 \quad \text{as} \quad n \to \infty$$

Now given an $\varepsilon > 0$ we may define

$$B = \bigcup_{k=1}^{\infty} B_{n_k}(k)$$ where n_k is so large

that $m(B_{n_k}(k)) < \varepsilon 2^{-k}$. Then

$$m(B) \leq \sum_{k=1}^{\infty} m(B_{n_k}(k)) < \varepsilon.$$ And for every $x \in E_\varepsilon = D \setminus B$

we have if $j > n_k$ then $|f_j(x) - f(x)| < \frac{1}{k}$. ∎

Since uniform convergence works well with continuity. Egoroff's then implies ~~That~~

Thm (Lusin): If f is integrable on D (say D is bounded), $\varepsilon > 0$, then $\exists g \in C(D)$ such that $m(\{x \in D;\ f(x) \neq g(x)\}) < \varepsilon$

Remark: Thus integrable functions are almost continuous.

Proof: f integrable $\Rightarrow \exists\, \varphi_j$ s.t.

i) φ_j is a simple function, $\varphi_{j+1} \geq \varphi_j$

ii) $\int_D |\varphi_j - f|\, dx \to 0$

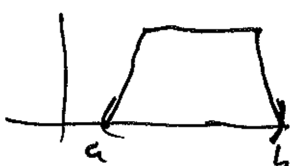
iii) $\varphi_j(x) \to f(x)$.

By Egoroff's theorem $\varphi_j \to f$ uniformly on some set $E_{\varepsilon/2}$, $m(D \setminus E_{\varepsilon/2}) < \varepsilon/2$.

Furthermore if

$$\varphi_j(x) = \sum_{k=1}^{N_j} a_k \chi_{A_k}$$

then we may approximate A_k by an open set, with error $\frac{\varepsilon}{8N_j}$, approximate the open set by finitely many M_j intervals with an error $\frac{\varepsilon}{8N_jM_j}$. On each intervall we may approximate φ_j by



continuous. This approximates φ_j by a continuous function with error $\varepsilon/8$, etc...