8.1| $x(n) = a_0 \sin(2\pi\nu_0 n + \Phi) + w(n)$ $\Phi \in [0, 2\pi)$ uniform

a_0, ν_0 ?

$R_w(k) = \sigma^2 \delta(k)$

$w(n)$ and Φ are independent.

$$\hat{P} = \frac{1}{N} \sum_{n=0}^{N-1} x^2(n) - \sigma^2$$

(a) $E[\hat{P}]$

$$E[\hat{P}] = \frac{1}{N} \sum_{n=0}^{N-1} E[x^2(n)] - \sigma^2$$

$$E[x^2(n)] \underset{P}{=} E[a_0^2 \sin^2(2\pi\nu_0 n + \Phi) + w^2(n)] =$$

Φ and $w(n)$ independent

and $w(n)$ has $\emptyset$ mean

$$= a_0^2 E[\sin^2(2\pi\nu_0 n + \Phi)] + \sigma^2 = \frac{a_0^2}{2\pi} \int_0^{2\pi} \sin^2(2\pi\nu_0 n + \Phi)\, d\Phi + \sigma^2 =$$

$$= \left\{ \begin{array}{l} -\cos(2\alpha) = -\cos^2(\alpha) + \sin^2(\alpha) \\ 1 = \cos^2(\alpha) + \sin^2(\alpha) \\ \hline 1 - \cos(2\alpha) = \quad / \quad + 2\sin^2(\alpha) \end{array} \right\} = \frac{a_0^2}{2\pi} \int_0^{2\pi} \left(\frac{1 - \cos(4\pi\nu_0 n + 2\Phi)}{2} \right) d\Phi$$

$$+ \sigma^2 = \frac{a_0^2}{4\pi} \left[\frac{1}{2}\Phi - 2\sin(4\pi\nu_0 n + 2\Phi) \right]_0^{2\pi} + \sigma^2 =$$

$$= \frac{a_0^2}{4\pi} \left[\frac{1}{2} 2\pi - \cancel{2\sin(4\pi\nu_0 n + 4\pi)} + \cancel{2\sin(4\pi\nu_0 n + \emptyset)} \right] + \sigma^2 =$$

$$= \frac{a_0^2}{4\cancel{\pi}} \frac{1}{2} \cancel{\pi} + \sigma^2 = \frac{a_0^2}{2} + \sigma^2$$

$$E[\hat{P}] = \frac{1}{N} \sum_{n=0}^{N-1} \left(\frac{a_0^2}{2} + \sigma^2 \right) - \sigma^2 = \frac{N}{N} \left(\frac{a_0^2}{2} + \cancel{\sigma^2} \right) \cancel{- \sigma^2}$$

$= \frac{a_0^2}{2}$ ⌋ The power of sinusoid is $\frac{a_0^2}{2}$ hence, its unbiased

(b) Φ deterministic but unknown

$E[\check{P}]$ for the corresponding estimator of this modified estimator

$$E[\hat{P}] = \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{E[x^2(n)]}_{\text{nothing stochastic in } x(n) \text{ except for the noise!}} - \sigma^2 =$$

$$= \frac{1}{N}\left[\sum_{n=0}^{N-1} \left(a_0^2 \sin^2(2\pi\nu_0 n + \Phi) + \sigma^2\right)\right] - \sigma^2 =$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} a_0^2 \cdot \sin^2(2\pi\nu_0 n + \Phi) = \frac{a_0^2}{N} \sum_{n=0}^{N-1} \sin^2(2\pi\nu_0 n + \Phi) =$$

$$= \frac{a_0^2}{N} \sum_{n=0}^{N-1} \frac{1}{2}\left(1 - \cos(4\pi\nu_0 n + 2\Phi)\right) = \frac{a_0^2}{2N} \sum_{n=0}^{N-1} 1 +$$

$$- \frac{a_0^2}{2N} \sum_{n=0}^{N-1} \cos(4\pi\nu_0 n + 2\phi) = \frac{a_0^2}{2} - \underbrace{\frac{a_0^2}{2N} \sum_{n=0}^{N-1} \cos(4\pi\nu_0 n + \phi)}_{\text{generally different from } 0}$$

→ unless special care is taken (by knowing before hand the value of ϕ) the estimator is biased.

c) $E[\check{P}]$? $N \to \infty$

$$E[\check{P}] = \frac{a_0^2}{2} - \frac{a_0^2}{2N} \sum_{n=0}^{N-1} \cos(4\pi\nu_0 n + 2\Phi)$$

$$\lim_{N\to\infty} \frac{a_0^2}{2} - \frac{a_0^2}{2N} \sum_{n=0}^{N-1} \cos(4\pi\nu_0 n + 2\phi) = \frac{a_0^2}{2}$$

<u>unbiased!</u> (Hint)

8.2] $X(t)$ be WSS with unknown mean m and with ACF:

$$r_X(\tau) = e^{-|\tau|} + m^2$$

- $\hat{m}_1 = \int_0^1 X(t)\,dt$ Bias, variance?
- $\hat{m}_2 = \frac{1}{2}\sum_{t=0}^{1} X(t)$ Bias, variance?

$\hat{m}_1$

- $E[\hat{m}_1] = E\left[\int_0^1 X(t)\,dt\right] = \int_0^1 E[X(t)]\,dt = \int_0^1 m\,dt =$

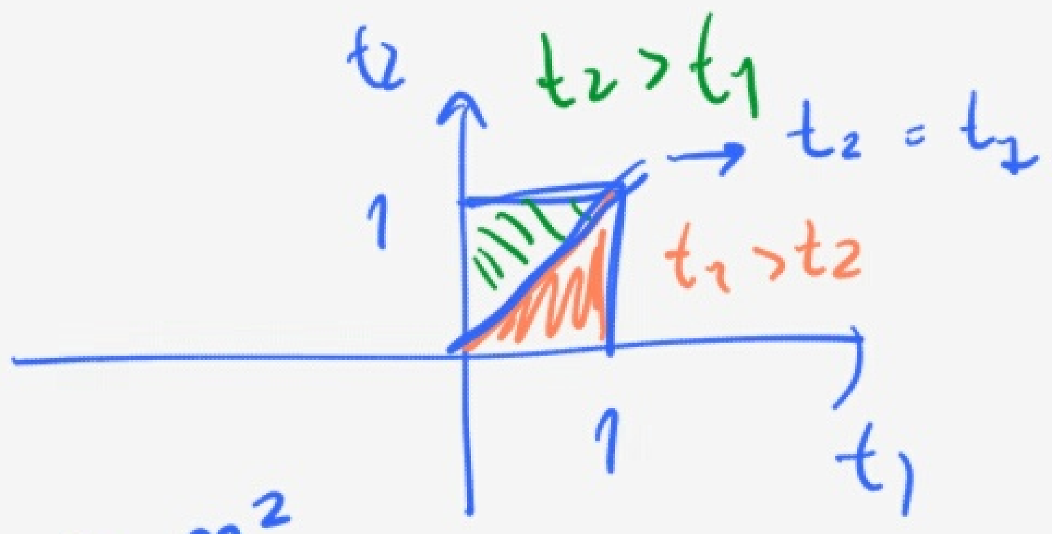
$m t\big]_0^1 = \underline{m}$

- $E[(\hat{m}_1 - m)^2] = E[\hat{m}_1^2] - m^2$

$$E[\hat{m}_1^2] = E\left[\int_0^1\int_0^1 X(t_1)X(t_2)\,dt_1\,dt_2\right] =$$

$$= \int_0^1\int_0^1 E[X(t_1)X(t_2)]\,dt_1\,dt_2 = \int_0^1\int_0^1 R_X(t_1,t_2)\,dt_1\,dt_2$$

$$= \text{WSS} = \int_0^1\int_0^1 \left(e^{-|t_2-t_1|} + m^2\right)dt_2\,dt_1 =$$



$$= \underbrace{\int_0^1 dt_1 \int_0^{t_1} e^{-|t_2-t_1|}}_{t_1 > t_2} + \underbrace{\int_0^1 dt_2 \int_0^{t_2} e^{-|t_2-t_1|}\,dt_1}_{t_2 > t_1} + m^2 =$$

$$= \int_0^1 e^{-t_1}\,dt_1 \int_0^{t_1} e^{t_2}\,dt_2 + \int_0^1 e^{-t_2}\,dt_2 \int_0^{t_2} e^{t_1}\,dt_2 + m^2 =$$

$$= 2\int_0^1 e^{-t_1}\,dt_1 \int_0^{t_1} e^{t_2}\,dt_2 + m^2 = 2\int_0^1 e^{-t_1}(e^{t_1}-1)\,dt_1 + m^2 =$$

$$= 2\int_0^1 (1 - e^{-t_1})\,dt_1 + m^2 = 2(t_1 + e^{-t_1})\big]_0^1 + m^2 =$$

$$= 2(1 + e^{-1}) - 2(\cancel{0} + e^0) + m^2 = 2e^{-1} + m^2$$

$$E[(\hat{m}_1 - m)^2] = 2e^{-1} + m^2 - m^2 = \underline{2e^{-1}}$$

$\hat{m}_2$:

- $E[\hat{m}_2] = E[\frac{1}{2}\sum_{t=0}^{1} X(t)] = \frac{1}{2}\sum_{t=0}^{1} E[X(t)] = \frac{1}{2}\sum_{t=0}^{1} m = \frac{2}{2} m$

unbiased.

- $E[(\hat{m}_2 - m)^2] = E[\hat{m}_2^{\,2}] - m^2$

$E[\hat{m}_2^2] = \frac{1}{4} E[\sum_{t_1=0}^{1} X(t_1) \sum_{t_2=0}^{1} X(t_2)] = \frac{1}{4}\sum_{t_1=0}^{1}\sum_{t_2=0}^{1} E[X(t_1)X(t_2)] =$

$= \frac{1}{4}\sum_{t_1=0}^{1}\sum_{t_2=0}^{1} R_X(t_1,t_2) = \left\{ \begin{matrix}\text{possible combinations are} \\ (0,0),(0,1),(1,0),(1,1)\end{matrix} \right\} =$

$= \frac{1}{4}(2R_X(0) + 2R_X(1)) = \frac{1}{4}(2e^0 + 2e^{-1}) =$

$= \frac{1}{2}(1 + e^{-1})$

9.51 $S(t)$ WSS with ACF $r_S(\tau)$ (Can also be done using normal equations directly)

$z = \int_0^T S(t)dt$ estimated using $\hat{z} = aS(0) + bS(T)$

a and B so that $\hat{z}$ is an MMSE estimate

$$\min E[(\hat{z}-z)^2] \Rightarrow \begin{matrix} E[(\hat{z}-z)S(0)] = 0 \\ E[(\hat{z}-z)S(T)] = 0 \end{matrix}$$

Error orthogonal to data!

- $E[\hat{z}S(0)] - E[zS(0)] = E[aS^2(0) + bS(T)S(0)]$

$-E[\int_0^T S(t)S(0)dt] = aR_S(0) + bR_S(T) - \int_0^T R_S(t)dt = \emptyset$

- $E[\hat{z}S(T)] - E[zS(T)] = aR_S(T) + bR_S(0) - \int_0^T R_S(T-t)dt = \emptyset$

Note! $\int_0^T R_S(T-t)dt = \int_0^T R_S(t)dt$

Proof: $T-t = a \Rightarrow \int_T^0 R_S(a)(-da) = \int_0^T R_S(a)\,da = \int_0^T R_S(t)dt$ ✓

$$\left.\begin{matrix} aR_S(0) + bR_S(T) = \int_0^T R_S(t)dt \\ aR_S(T) + bR_S(0) = \int_0^T R_S(t)dt \end{matrix}\right\} \text{solve for } a \text{ and } b: \quad a = b = \frac{\int_0^T r_S(t)dt}{r_S(0) + r_S(T)}$$