Set partitions

- Let S be a finite set. A <u>partition</u> of S into k blocks is an (unordered) collection $\{B_1, \dots, B_k\}$ of subsets of S s.t.

  (1): $\emptyset \subsetneq B_i \subseteq S \quad \forall i \in [k]$

  (2): $B_i \cap B_j = \emptyset \quad \forall i \neq j$

  (3): $\bigcup_{i=1}^{k} B_i = S$

- Let $S(n,k)$ be the number of partitions of $[n]$ into k blocks. $\{S(n,k)\}$ are called the "Stirling numbers of the 2nd kind."

- $S(n,0)=0$

  $S(n,1)=1$

  $S(n,2)=2^{n-1}-1$

  $\vdots$

- <u>Prop.</u> $S(n,k) = kS(n-1,k) + S(n-1,k-1)$

  <u>Proof</u>: Either

  (a). n is in a block with other elements: $kS(n-1,k)$, or

  (b). in a singleton block; $S(n-1,k-1)$. $\square$

- Recall the "falling factorials"

  $(x)_0 = 1$

  $(x)_1 = x$

  $(x)_2 = x(x-1)$

  $(x)_k = x(x-1)\cdots(x-k+1)$

- Hence $(x)_k = k!\binom{x}{k}$

Prop: $$x^n = \sum_{k=0}^{n} S(n,k)(x)_k = \sum_{k=0}^{n} S(n,k)k!\binom{x}{k}$$ 26

Proof: Note that is enough to prove the identity for all $x = m$, where m is a positive number (Why?).

The number of surjections $f: [n] \to [k]$ is equal to $k!\,S(n,k)$, since given a partition of $[n]$ into k blocks we may associate exactly $k!$ surjections by determining which block is the inverse image of which integer.

Each function $f: [n] \to [m]$ is a surjection onto a unique subset of $[m]$;

$$m^n = \sum_{S \subseteq [m]} |\{\text{surjections } g: [n] \to S\}|$$

$$= \sum_{k=0}^{n} k!\,S(n,k)\binom{m}{k} \qquad \square$$

size of S $\to$ $k=0$

· Bell numbers: $B(n) = \#$ partitions of $[n]$

$$= \sum_{k=0}^{n} S(n,k). \quad (B(0) := 1)$$

· $$B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i) = \sum_{i=0}^{n} \binom{n}{i} B(n-i)$$

$\uparrow$ which elements are in the same block as $n+1$.

· Exercise: $$\sum_{n=0}^{\infty} B(n)\frac{x^n}{n!} = \exp\left(e^x - 1\right).$$

- We will see later that

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

- Exercise: $\displaystyle\sum_{n \geq 0} S(n,k) \frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k$

- Note that both $(X)_0, (X)_1, \ldots, (X)_N$ and $1, X, X^2, \ldots, X^N$ are bases for the vector space $V_N = \{ f \in \mathbb{C}[X] : \deg f \leq N \}$

- The (signed) Stirling number of the first kind is $s(n,k) = (-1)^{n-k} c(n,k)$.

- The two identities:

$$\sum_{k=0}^{n} s(n,k) X^k = (-1)^n \sum_{k=0}^{n} c(n,k)(-X)^k = (-1)^n (-X)(-X+1)\ldots(-X+n-1) = (X)_n$$

and

$$\sum_{k=0}^{n} S(n,k)(X)_k = X^n$$

implies that the two matrices $\left[S(n,k) \right]_{k,n=0}^{N}$ and $\left[s(n,k) \right]_{k,n=0}^{N}$ are inverses of each other.

Finite differences

- Let $V = \mathbb{C}^{\mathbb{N}} = \{ f : \mathbb{N} \to \mathbb{C} \}$. Consider two linear operators

$$\Delta : V \to V$$
$$E : V \to V$$

$\Delta(f)(n) = f(n+1) - f(n)$ "first difference operator"

$E(f)(n) = f(n+1)$ "shift operator"

- Note that $\Delta = E - I$, where I is the identity operator $I(f) = f$. Hence

- Let $\Delta^k = \underbrace{\Delta \circ \Delta \circ \dots \circ \Delta}_{k \text{ times}}$, $\Delta^0 = I$

$$E^k = \underbrace{E \circ \dots \circ E}_{k \text{ times}}.$$

- Then, for $n \in \mathbb{N}$,

$$f(n) = E^n(f)(0) = [(\Delta + 1)^n f](0)$$
$$= \sum_{k=0}^{n} \binom{n}{k} (\Delta^k f)(0) = \sum_{k=0}^{n} \Delta^k(f)(0) \cdot \binom{n}{k}$$

and

$$\Delta^n(f)(k) = (E - I)^n(k) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (E^j f)(k)$$

(set $k=0$)

$$\therefore \begin{cases} f(n) = \sum_{k=0}^{n} \Delta^k(f)(0) \cdot \binom{n}{k} \\ \Delta^n(f)(0) = \sum_{k=0}^{n} (-1)^{n-k} f(k) \binom{n}{k} \end{cases}$$

Recall $$x^n = \sum_{k=0}^{n} k!\,S(n,k)\binom{x}{k}$$

Hence $$\Delta^k(x^n)(0) = k!\,S(n,k) = \sum_{k=0}^{n} (-1)^{n-k} k^n \binom{n}{k}$$

- Note that for fixed k:

$$\Delta\binom{n}{k} = \binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1}$$

$$\Delta n^k = (n+1)^k - n^k = n^k + k n^{k-1} + \binom{k}{2} n^{k-2} + \dots + 1 - n^k$$
$$= \text{polynomial of deg} = k-1.$$

<u>Prop.</u> Let $f \in V$.

(a). f is a polynomial of deg $\leq d \iff \Delta^{d+1}(f) \equiv 0$

(b). If (a), then
$$f(n) = \sum_{k=0}^{d} \Delta^k f(0)\binom{n}{k}$$

- How do we compute $\Delta^k f(0)$?

Take $f(n) = n^4$

$$\begin{array}{ccccccccccc}
0 & & 1 & & 16 & & 81 & & 256 & & 625 \ \dots \\
& 1 & & 15 & & 65 & & 175 & & 369 & \\
& & 14 & & 50 & & 110 & & 194 & & \\
& & & 36 & & 60 & & 84 & & & \\
& & & & 24 & & 24 & & & & \\
& & & & & 0 & & & & &
\end{array}$$

$$\Delta^k(n^4)(0) = (0,1,14,36,24,0) = (S(4,k)\cdot k!)$$

<u>Corollary</u>: Let G be the abelian group of all polynomials $f : \mathbb{N} \to \mathbb{Z}$ of degree at most d. Then G is free with basis $\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{d}$.

Sieve Methods

- Principle of Inclusion-Exclusion.

Recall $|A \cup B| = |A| + |B| - |A \cap B|$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\substack{T \subseteq [n] \\ |T| \geq 1}} (-1)^{|T|-1} \left|\bigcap_{i \in T} A_i\right| \qquad (*)$$

Theorem. Let V be a finite set and let $f, g : 2^V \to \mathbb{C}$ be two functions. TFAE

(a). $g(S) = \sum_{T \supseteq S} f(T)$, $\forall\, S \subseteq V$

(b) $f(S) = \sum_{T \supseteq S} (-1)^{|T-S|} g(T)$, $\forall\, S \subseteq V$.

Proof: May assume $V = [n]$ for some $n > 0$. Let

$$F(x_1, \dots, x_n) = \sum_{T \subseteq [n]} f(T) \prod_{i \in T} x_i$$

$$G(x_1, \dots, x_n) = \sum_{T \subseteq [n]} g(T) \prod_{i \in T} x_i$$

Then

$$F(x_1+1, \dots, x_n+1) = \sum_T f(T) \prod_{i \in T} (x_i + 1)$$

$$= \sum_T f(T) \sum_{S \subseteq T} \prod_{i \in S} x_i$$

$$= \sum_S \Big(\sum_{T \supseteq S} f(T) \Big) \prod_{i \in S} \qquad \text{and}$$

$$G(x_1-1, \dots, x_n-1) = \sum_T g(T) \prod_{i \in T} (x_i - 1) = \sum_T g(T) \sum_{S \subseteq T} (-1)^{|T-S|} \prod_{i \in S} x_i$$

$$= \sum_S \Big(\sum_{T \supseteq S} (-1)^{|T-S|} g(T) \Big) \prod_{i \in S} x_i$$

Hence (a) holds iff $F(x_1+1, \dots, x_n+1) = G(x_1, \dots, x_n)$, and (b) holds iff $G(x_1-1, \dots, x_n-1) = F(x_1, \dots, x_n)$

$\therefore$ (a) $\Leftrightarrow$ (b) $\square$

Equivalent form: (a)' and (b)' below are equivalent:

$$(a)': \quad g(T) = \sum_{S \subseteq T} f(S), \quad \forall\, T \subseteq V$$

$$(b)': \quad f(T) = \sum_{S \subseteq T} (-1)^{|T \setminus S|} g(S), \quad \forall\, T \subseteq V.$$

Exercise: Deduce $(*)$.

If f and g only depend on $|S|$, we get the following corollary:

Corollary: Let $f, g : [n] \to \mathbb{C}$. TFAE

$$(a). \quad b(m) = \sum_{i=0}^{m} \binom{m}{i} a(i), \quad 0 \le m \le n$$

$$(b). \quad a(m) = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} b(i), \quad 0 \le m \le n.$$

- Hence the matrices $\left[\binom{m}{i}\right]_{i,m=0}^{n}$ and $\left[(-1)^{m-i}\binom{m}{i}\right]_{i,m=0}^{n}$ are inverses of each other.
- Note that $a(m) = (\Delta^m b)(0)$, $0 \le m \le n$.

Example: Derangements.

A permutation $\pi \in \mathfrak{S}_n$ is a derangement if it has no fixed points. Let $D(n)$ denote the number of derangements in $\mathfrak{S}_n$:

$D(0) := 1$, $D(1) = 0$, $D(2) = 1$, $D(3) = 2$, ...

The number of permutations in $\mathfrak{S}_m$ with exactly k fixed points is

$$\underbrace{\binom{m}{k}}_{\text{choose which elements are fixed}} \underbrace{D(m-k)}_{\text{The rest forms a derangement on } m-k \text{ letters}}$$

$$\therefore \quad m! = \sum_{k=0}^{m} D(m-k)\binom{m}{k} = \sum_{k=0}^{m} D(k)\binom{m}{k}$$

By the corollary:

$$D(m) = \sum_{i=0}^{m} (-1)^{m-i} i! \binom{m}{i} = \sum_{i=0}^{m} (-1)^{m-i} \frac{m!}{(m-i)!}$$

$$= m! \sum_{j=0}^{m} \frac{1}{j!}(-1)^j$$

. We can phrase this as the probability that a random m-permutation is a derangement is $\dfrac{D(m)}{m!} = \sum_{j=0}^{m} \frac{(-1)^j}{j!}$

Note that $\lim_{m\to\infty} \dfrac{D(m)}{m!} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} = e^{-1} = 0.367879\ldots$