ENUMERATIVE COMBINATORICS (SF2708), 2016
PROBLEM SET 1

The problems are due **October 11**. You may discuss the problems with other students in the class, but with no one else. You may not copy solutions you have found elsewhere. If you discuss with other student(s) in the class you should mention the name(s) for each problem. Each and every one should write down your own solution in your own words. Maximal credit will be given only to complete and clear solutions.

(1) Let M_n be the set of permutations $\pi \in \mathfrak{S}_n$ such that for all $2 \leq i \leq n$ there exists $1 \leq j < i$ with $|\pi(i) - \pi(j)| = 1$. For example, $M_3 = \{123, 213, 231, 321\}$. Furthermore, for $S \subseteq [n-1]$, let $M_n(S)$ be the set of permutations in M_n with descent set S. Determine $|M_n(S)|$ and $|M_n|$.

(2) Let F_n be the set of all $2 \times n$ matrices with entries 0 or 1 such that no 1's are adjacent in columns or in rows. Let further $F_{n,k}$ be the set of all such matrices with exactly k ones. Determine $|F_n|$ and $|F_{n,k}|$.

(3) Problem 1.12 in EC1. *Hint:* One may use Euler's formula for planar graphs.

(4) Problem 1.26 in EC1.

(5) Problem 1.54 in EC1.

(6) Problem 1.102ab in EC1.

(7) For positive integers n and d, let $\mathcal{A}_{d,n}$ be the set of all pairs (f, r) of functions $f : [d] \to [n]$ and $r : [d] \to [d]$ satisfying
   - f is weakly increasing, i.e., if $1 \leq i \leq j \leq d$, then $f(i) \leq f(j)$,
   - $r(i) \leq i$, for all $i \in [d]$,
   - if $f(i) = f(j)$ for some $i \neq j$, then $r(i) < r(j)$.

   Prove that $|\mathcal{A}_{d,n}| = n^d$.

(8) Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a vector of variables, and for $k = 0, \ldots, n$, let

$$e_k(\mathbf{x}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k} = \sum_{\substack{S \subseteq [n] \\ |S| = k}} \prod_{j \in S} x_j$$

be the elementary symmetric polynomials. Prove combinatorially that

$$\sum_{k=0}^{n} e_k(\mathbf{x})^2 = x_1 \cdots x_n \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{2j}{j} e_{n-2j}\left(x_1 + \frac{1}{x_1}, \ldots, x_n + \frac{1}{x_n}\right).$$

For $n = 2$ the identity reads

$$1 + (x_1 + x_2)^2 + x_1^2 x_2^2 = x_1 x_2 \left(\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right) + 2\right)$$