9.1.1 X and Y stochastic with $m_y = 4$, $\sigma_y = 1$, $m_x = 0$, $\sigma_x = 2$ and $E[XY] = 1$

a) Determine a that minimizes $E\{(X - aY)^2\}$ and the corresponding minimum value.

error, hence X parameter to estimate, a "filter", Y data

orthogonality principle

$(E\{(X-aY)Y\} = 0)$ $\quad E\{XY\} - aE\{Y^2\} = 0$

$1 - a(\sigma_y^2 + m_y^2) = 0 \quad 1 - a(1+16) = 0 \quad 1 - a17 = 0 \quad \boxed{a = \frac{1}{17}}$

$$\varepsilon = E\{(X-aY)^2\}\Big|_{a=\frac{1}{17}} = E\{(X - \tfrac{1}{17}Y)^2\} = E\{X^2 - 2\tfrac{1}{17}XY + \tfrac{1}{(17)^2}Y^2\} =$$

$$= (\sigma_x^2 + m_x^2) - 2\frac{1}{17}E[XY] + \frac{1}{(17)^2}(\sigma_y^2 + m_y^2) = 4 - \frac{2}{17} + \frac{1}{(17)^2}(17) =$$

$$= 4 - \frac{2}{17} + \frac{1}{17} = \frac{68 - 2 + 1}{17} = \boxed{\frac{67}{17}}$$

b) Constants a_0 and a_1 that minimize $E\{(X - (a_0 + a_1 Y))^2\}$

orthogonality principle: $\quad \underline{a} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \quad \underline{d} = \begin{pmatrix} 1 \\ Y \end{pmatrix}$

$$E\{\underbrace{(X - \underline{a}^T\underline{d})}_{\text{error}}\underbrace{\underline{d}^T}_{\text{data}}\} = \underline{0}^T$$

$$E\{X\underline{d}^T\} - E\{\underline{a}^T\underline{d}\,\underline{d}^T\}$$

$$E\{X(1, Y)\} - \underline{a}^T E\left\{\begin{pmatrix} 1 \\ Y \end{pmatrix}(1\ Y)\right\} =$$

$$= (m_x, E[X,Y]) - \underline{a}^T\begin{pmatrix} 1 & m_y \\ m_y & \sigma_y^2 + m_y^2 \end{pmatrix} = 0 \quad \Rightarrow \text{Take transpose}$$

$$\begin{pmatrix} m_x \\ E[X,Y] \end{pmatrix} - \begin{pmatrix} 1 & m_y \\ m_y & \sigma_y^2 + m_y^2 \end{pmatrix}\underline{a} = 0$$

$$\underline{a} = \begin{pmatrix} 1 & m_y \\ m_y & \sigma_y^2 + m_y^2 \end{pmatrix}^{-1}\begin{pmatrix} m_x \\ E[X,Y] \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 17 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 17 & 4 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 4 & 17 \end{pmatrix}^{-1} = \frac{\begin{pmatrix} 17 & -4 \\ -4 & 1 \end{pmatrix}}{17 - 16} = \begin{pmatrix} 17 & -4 \\ -4 & 1 \end{pmatrix}$$

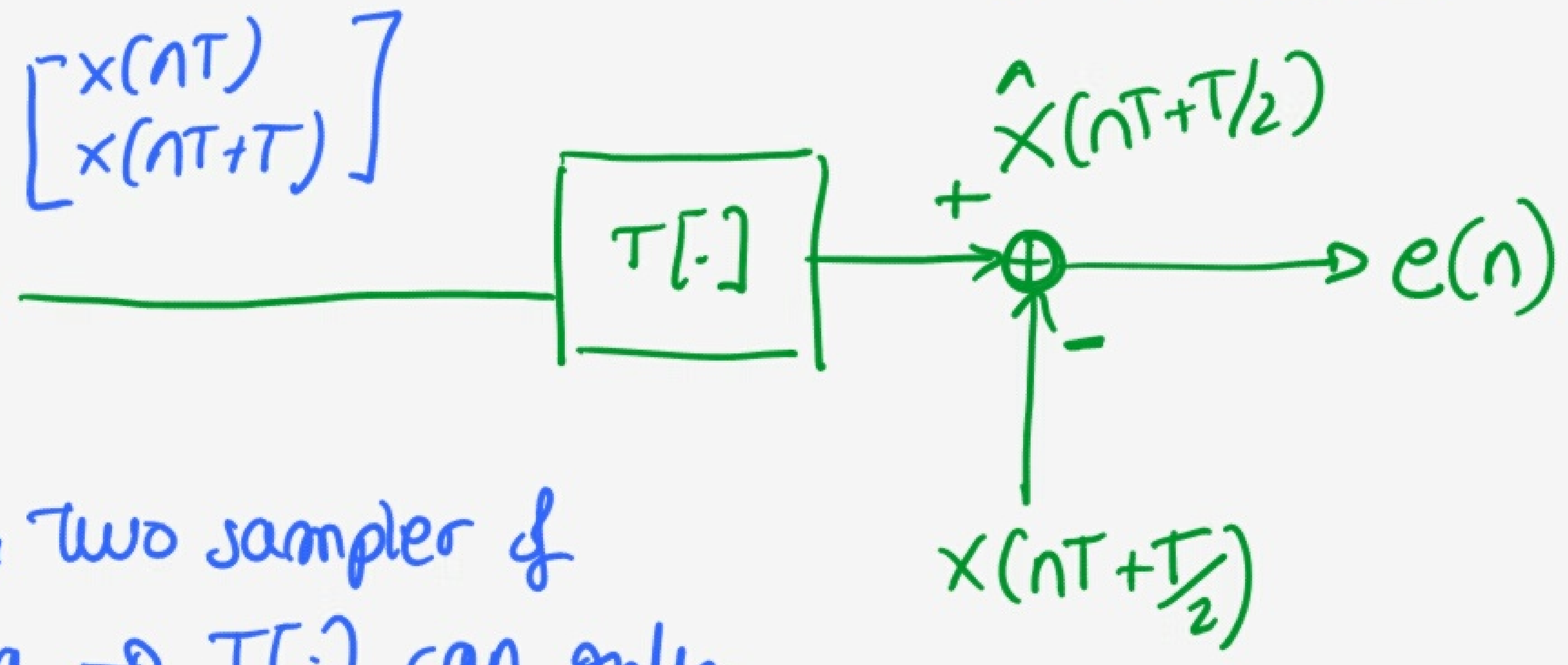
Hence, $\underline{a} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \Rightarrow a_0 = -4,\ a_1 = 1$

9.7.

$x(t)$ Realisation of $X(t)$ with ACF $r_x(\tau)$

sampled signal $x(nT)$.

Try to estimate $x(nT+T/2)$ determining a linear combination of $x(nT)$ and $x(nT+T)$ which minimizes MMSE between interpolated value and the true value.



only two samples of data $\Rightarrow$ T[·] can only have 2 coefficients!

$$E\{(e(n))\,[x(nT), x(nT+T)]\} = \underline{0}^T$$

$$e(n) = \hat{x}(nT+T/2) - x(nT+T/2) =$$
$$= [h_1, h_2]\begin{bmatrix} x(nT) \\ x(nT+T) \end{bmatrix} - x(nT+T/2)$$

$$E\left\{\left([h_1, h_2]\begin{bmatrix} x(nT) \\ x(nT+T) \end{bmatrix} - x(nT+T/2)\right)[x(nT), x(nT+T)]\right\} = \emptyset$$

$$E\{(\underline{h}^T\underline{x} - x(nT+T/2))\underline{x}^T\} = 0$$

$$\underline{h}^T E\{\underline{x}\underline{x}^T\} - E\{x(nT+T/2)\underline{x}^T\} = \emptyset$$

$$\underline{h}^T E\left\{\begin{bmatrix} x(nT) \\ x(nT+T) \end{bmatrix}[x(nT)\; x(nT+T)]\right\} - E\{x(nT+T/2)\,[x(nT)\; x(nT+T)]\}$$
$$= \emptyset$$

$$\underline{h}^T\begin{pmatrix} R_x(0) & R_x(T) \\ R_x(T) & R_x(0) \end{pmatrix} - \begin{pmatrix} R_x(T/2) \\ R_x(T/2) \end{pmatrix}^T = \emptyset \Rightarrow \text{take transpose}$$

$$\begin{pmatrix} R_x(0) & R_x(T) \\ R_x(T) & R_x(0) \end{pmatrix}\underline{h} - \begin{pmatrix} R_x(T/2) \\ R_x(T/2) \end{pmatrix} = 0$$

$$\underline{h} = \begin{pmatrix} R_x(0) & R_x(T) \\ R_x(T) & R_x(0) \end{pmatrix}^{-1}\begin{pmatrix} R_x(T/2) \\ R_x(T/2) \end{pmatrix}$$

$$\underline{h} = \begin{pmatrix} R_x(0) & R_x(T) \\ R_x(T) & R_x(0) \end{pmatrix}^{-1} \begin{pmatrix} R_x(T/2) \\ R_x(T/2) \end{pmatrix}$$

$$\underline{h} = \frac{\begin{pmatrix} R_x(0) & -R_x(T) \\ -R_x(T) & R_x(0) \end{pmatrix}}{R_x(0)^2 - R_x(T)^2} \begin{pmatrix} R_x(T/2) \\ R_x(T/2) \end{pmatrix}$$

$h_1 = h_2$ (symmetry of matrix and vector being all same)

$$h_1 = h_2 = \frac{R_x(0)R_x(T/2) - R_x(T)R_x(T/2)}{(R_x(0) + R_x(T))\cancel{(R_x(0) - R_x(T))}} = \boxed{\frac{R_x(T/2)}{R_x(0) + R_x(T)}}$$

- a straight line interpolation can be expressed as:

$$\hat{x}(nT + T/2) = \frac{1}{2}(x(nT) + x(nT+T))$$

- mmse interpolation

$$\hat{x}_{MMSE}(nT + T/2) = \frac{R_x(T/2)}{R_x(0) + R_x(T)}(x(nT) + x(nT+T))$$

9.18 Let $X(n)$ and $Y(n)$ be two stochastic processes. $X(n)$ given by

$$\hat{X}(n) = h(n) * Y(n) = \sum_{k=-\infty}^{\infty} h(k)Y(n-k)$$

Estimator to be a Wiener filter should be chosen such that the MSE $E\{(X(n) - \hat{X}(n))^2\}$ is minimized.

a) Show that the orthogonality principle gives the following necessary condition on the Wiener filter.

$$E\{(X(n) - \hat{X}(n))^2\} = E\{(X(n) - \sum_{k=-\infty}^{\infty} h(k)Y(n-k))^2\}$$

$$\nabla_{\underline{h}} E\{(X(n) - \sum_{k=-\infty}^{\infty} h(k)Y(n-k))^2\} = \emptyset$$

∞

given $h(a)$ we have that $\forall a$:

$$\frac{d}{dh(a)} E\{(X(n) - \sum_{k=-\infty}^{\infty}{}' h(k)Y(n-k))^2\} = 0$$

$$\frac{d}{dh(a)} E\{\cdot\} = \frac{d}{dh(a)} E\{(X(n))^2 - 2\sum_{k=-\infty}^{\infty}{}' h(k)Y(n-k)X(n) + \sum_{k=-\infty}^{\infty}{}' \sum_{\ell=-\infty}^{\infty}{}' h(k)h(\ell)Y(n-\ell)Y(n-k)\}$$

$$= \frac{d}{dh(a)} E\{X^2(n) - 2h(a)Y(n-a)X(n) - 2\sum_{k\neq a}{}' h(k)Y(n-k)X(n) + h^2(a)Y(n-a)Y(n-a)$$

$$+ Y(n-a)h(a)\sum_{\ell\neq a}{}' h(\ell)Y(n-\ell) + Y(n-a)h(a)\sum_{k\neq a}{}' h(k)Y(n-k) + \sum_{\ell\neq a}{}'\sum_{k\neq a}{}' h(\ell)h(k)Y(n-k)Y(n-\ell)\}$$

$$= -2E[Y(n-a)X(n)] + 2h(a)E[Y(n-a)Y(n-a)] + 2\sum_{\ell\neq a}{}' h(\ell)E[Y(n-\ell)\cdot Y(n-a)] = 0$$

$$\Rightarrow E[Y(n-a)X(n)] - \sum_{\ell=-\infty}^{\infty}{}' h(\ell)E[Y(n-a)Y(n-\ell)] = 0$$

$$E[(X(n) - \sum_{k=-\infty}^{\infty}{}' h(k)Y(n-k))Y(n-a)] = 0$$

holds $\forall n, \forall a$. Let $\ell \triangleq n-a$ then $\forall \ell$:

$$E[(X(n) - \sum_{k=-\infty}^{\infty}{}' h(k)Y(n-k))Y(n-a)] = 0$$

b) Show that the frequency function $H(\nu) = \mathcal{F}\{h(k)\} = \sum_{k=-\infty}^{\infty}{}' h(k)e^{-j2\pi\nu k}$ of the Wiener filter is $H(\nu) = \frac{R_{xy}(\nu)}{R_y(\nu)}$

$$E\{(X(n) - \sum_{k=-\infty}^{\infty}{}' h(k)Y(n-k))Y(\ell)\} = \emptyset$$

$$E\{X(n)Y(\ell)\} - \sum_{k=-\infty}^{\infty}{}' h(k)E\{Y(n-k)Y(\ell)\} = \emptyset \qquad \forall n, \forall \ell$$

$n-\ell =$

$$R_{XY}(n-\ell) - \sum_{k=-\infty}^{\infty}{}' h(k) R_y(n-k-\ell) = \emptyset$$

$$R_{xy}(n-\ell) - \sum_{k=-\infty}^{\infty} h(k)\, R_y(n-k-\ell) = R_{xy}(n-\ell) - h(n) * R_y(n-\ell)$$

$$\{n-\ell = m\} \rightarrow \quad R_{xy}(m) - \sum_{k=-\infty}^{\infty} h(k)\, R_y(m-k) = R_{xy}(m) - h(m) * R_y(m)$$

$= \emptyset$

We take Fourier transform on both sides $\Rightarrow R_{xy}(\nu) - H(\nu)\, R_y(\nu) = \emptyset$

$$H(\nu) = \frac{R_{xy}(\nu)}{R_y(\nu)}$$