Product measures and Fubini.

When we defined m we began by defining

$$m^*(A) = \inf \sum_{j=1}^{\infty} (b_j - a_j) \qquad \text{where } A \subset \bigcup_{j=1}^{\infty} (a_j, b_j).$$

Then we restricted m^* to all sets A such that

$$m^*(X) = m^*(X \cap A) + m^*(X \cap A^c).$$

If we want to calculate the area, volume, etc. of a set $A \subset \mathbb{R}^n$ we may reason similarly.

1) If $A = A_1 \times A_2 \times \dots \times A_n$ where A_j is measurable in $\mathbb{R}$.

then we can define $m^*_{\mathbb{R}^n}(A) = m(A_1) \times m(A_2) \cdots m(A_n)$

2) For any set A define

$$m^*_{\mathbb{R}^n}(A) = \inf \sum_{j=1}^{\infty} m(A_1^j) \cdots m(A_n^j)$$

where $A \subset \bigcup_{j=1}^{\infty} A_1^j \times \dots \times A_n^j$ ~~and~~ Disjoint!

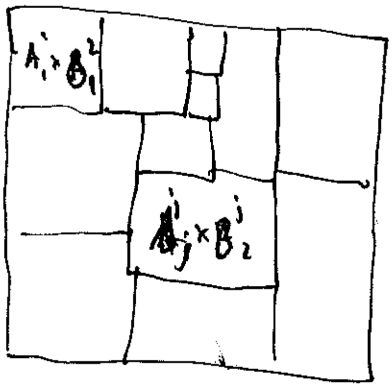
Notice that we have two definitions of $m(A)$ for A a rectangle.

<u>Lemma</u>: (Only $\mathbb{R}^2$) If ~~A_2~~ $A_1 \times B_2$ ~~then~~ and

$$A_1 = \bigcup A_1^j \times B_2^j \qquad (\text{disjoint}) \qquad \text{then}$$

$$m(A_1) \times m(B_2) = \sum_{j=1}^{\infty} m(A_1^j)\, m(B_2^j).$$

Proof:



A_1 B_2

Observe that we may write

$$B = \bigcup_{j,\, x \in A_j} B_j$$

for any $x \in A$

Since the Lebesgue measure is countably additive

$$m(B) = m\Big(\bigcup_{j,\, x\in A_j} B_j\Big) = \sum_{j,\, x \in A_j} m(B_j)$$

We may therefore write

$$m(B)\chi_A(x) = \sum_{j=1}^{\infty} m(B_j)\,\chi_{A_j}(x)$$

Now A_j are measurable wherefore χ_{A_j} is integrable and thus $\lim_{N\to\infty} \sum_{k=1}^{N} m(B_j)\chi_{A_j}(x) \leq m(B)\chi_A$ is integrable by the monotone convergence thm.

We may conclude that

$$m(A)\times m(B) = \int m(B)\chi_A \, dm = \sum_j \int m(B_j)\chi_{A_j} \, dm = \sum_{j=1}^{\infty} m(A_j)\times m(B_j)$$

~~We can~~

It follows that $m^*(A\times B)$ takes the value we would want.

Finally we define, in analogy with the 1d case,

<u>Definition</u>: We say that $S\subset\mathbb{R}^n$ is measurable if for all $X\subset\mathbb{R}^n$

$$m^*(X) = m^*(X\cap S) + m^*(X\cap S^c).$$

The restriction of m^* to all measurable sets will be denoted m.

Before we state the Fubini theorem, which is the main theorem of this lecture, we need the following result:

<u>Thm</u>: If we denote by $S_\sigma = \{\bigcup_{j=1}^{\infty} A_j\times B_j \,;\, A_j \,\&\, B_j \text{ measurable}\}$ and $S_{\sigma\delta} = \{\bigcap_{j=1}^{\infty} T_j \,;\, T_j\in S_\sigma\}$.

Then for any set $E\subset\mathbb{R}^2$ there exists $A\subset S_{\sigma\delta}$ s.t.

$E\subset A$ and $m^*(A) = m^*(E)$.

If E is measurable then $m^*(A\setminus E) = 0$

<u>Fubini Theorem:</u> Assume that $f(x,y)$ is integrable (measurable and $\iint_{\mathbb{R}^2} |f| < \infty$) over $\mathbb{R}^2$ with respect to $m_{\mathbb{R}^2}$. Then

a) for a.e. $x \in \mathbb{R}$, $f(x,\cdot)$ is integrable over $\mathbb{R}$

b) $$\int_{\mathbb{R}^2} f(x,y)\, dm_{\mathbb{R}^2} = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x,y)\, dm(y) \right] dm(x).$$

<u>Remark:</u> If ~~$A\times B$~~ E is measurable then we may interpret $\int_E f(x,y)\, dm_{\mathbb{R}^2}(x,y) = \int_{\mathbb{R}^2} f \chi_E \, dm_{\mathbb{R}^2}$

The proof is rather complicated so we will do it in several steps

1) We begin by showing the theorem for $f = \chi_E$ where E is an $S_{\sigma\delta}$ set.

2) Theorem holds for $f = \chi_E$ if E is a zero set

3) S measurable $\Rightarrow$ $S = (S_{\sigma\delta} \text{ set}) \cup (\text{zero set})$ so the Theorem holds for $f = \chi_S$ S measurable

4) Theorem holds for simple functions $\varphi = \sum a_i \chi_{S_i}$

5) For general f we may approximate f ~ simple and pass to the limit.

Lemma: Let E be an $S_{\sigma\delta}$ set and $m_{\mathbb{R}^2}(E) < \infty$ then

a) $E_x = \{y;\ (x,y) \in E\}$ is measurable

b) $x \longmapsto m(E_x)$ is a measurable function

c) $$m^*_{\mathbb{R}^2}(E) = \int_{\mathbb{R}} (m(E_x))\, dm(x).$$

Proof: That the Lemma holds for S_σ sets follows as the proof that $m^*_{\mathbb{R}^2}$ is well defined on rectangles.

Therefore we only need to show it for $S_{\sigma\delta}$ sets we may assume that

$$E = \bigcap_k E_k \quad \text{where} \quad E_k \in S_\sigma$$

Since $m_{\mathbb{R}^2}(E) < \infty$ then we may assume that $m_{\mathbb{R}^2}(E_1) < \infty$

Since E_k is a S_σ set $(E_k)_x$ is measurable and $\left(\bigcap_{k=1}^N E_k\right)_x$ is the intersection of measurable sets, send $N \to \infty$ to derive that $(E)_x = \left(\bigcap_{k=1}^\infty E_k\right)_x$ is measurable.

Also, by the continuity of m

$$m((E)_x) = \lim_{k\to\infty} m((E_k)_x)$$

Since $x \mapsto m((E_k)_x)$ is measurable for each k the dominated convergence thm implies that

$$\int_{\mathbb{R}} m(E_x)\, dm(x) = \lim_{k\to\infty} \int_{\mathbb{R}} m((E_k)_x)\, dm = \lim m_{\mathbb{R}^2}(E_k).$$

It remains to show that $\lim_{k\to\infty} m_{\mathbb{R}^2}(E_k) = m_{\mathbb{R}^2}(E_k)$ but that follows as the same proof for the Lebesgue measure on $\mathbb{R}$.

$\blacksquare$

<u>Lemma</u>: If $m_{\mathbb{R}^2}(E) = 0$ then

a) for almost all x, E_x is measurable

and $m(E_x) = 0$

Thus $m_{\mathbb{R}^2}(E) = \int_{\mathbb{R}} \underbrace{m(E_x)}_{=0 \text{ a.e}}\, dm(x) = 0.$

<u>Proof:</u> We may find $A \subset S_{\sigma\delta}$ s.t. $E \subset A$ $m_{\mathbb{R}^2}(A) = 0$. By the previous Lemma

$$0 = m_{\mathbb{R}^2}(A) = \int_{\mathbb{R}} m(A_x)\, dx \implies m(A_x) = 0 \quad \text{a.e. } x$$

and since $E \subset A \implies E_x \subset A_x \implies m(E_x) = 0.$

<u>Proposition</u>: E measurable $\Rightarrow$ i) E_x is measurable a.e. x

ii) $x \to m(E_x)$ is measurable

$$m_{\mathbb{R}^2}(E) = \int_X m(E_x)\, dm(x)$$

<u>Proof</u>: ~~Write~~ $E \subset A$ where $A \subset S_{\sigma\delta}$ and $m_{\mathbb{R}^2}(A \setminus E) = 0$. Use previous lemmas on A and

$$m(A_x) = m(E_x) + \underbrace{m(A_x \setminus E_x)}_{=0 \text{ a.e. } x} = m(E_x) \quad \text{a.e. } x.$$

<u>Thm</u>: If $f = \sum_{j=1}^{N} a_j \chi_{E_j}(x)$ is a simple function (in particular E_j are measurable)

Then
$$\int_{\mathbb{R}^2} f \, dm_{\mathbb{R}^2} = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f \, dm(y) \right] dm(x).$$

and $f(x, \cdot)$ is integrable in y for a.e. x.

<u>Proof</u>: Linearity of the integral. $\square$

Proof of the Fubini Theorem:

No loss of generality to assume that $f(x,y) \geq 0$

Let φ_k be simple functions $\varphi_k = \sum_{j=1}^{k} \varepsilon_k j \, \chi_{\{E_j\}}$,

$\varepsilon_k = \frac{1}{\sqrt{k}}$ (or whatever)

$E_j = \{(x,y);\ \varepsilon j < f(x,y) \leq \varepsilon(j+1)\}$

Then

$$\int_{\mathbb{R}^2} \varphi_k \, dm_{\mathbb{R}^2} = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \varphi_k(x,y) \, dm(y) \right] dm(x)$$

and $\varphi_k \to f$ ~~a.e.~~ pointwise

We need to show that

a) for a.e. x $f(x,\cdot)$ is integrable.

But each φ_k is integrable for a.e. y

Say $\varphi_k(x,\cdot)$ is integrable on $\mathbb{R} \setminus N_k$ where $m(N_k) = 0$. Then for each $x \in \mathbb{R} \setminus \underbrace{\left(\bigcup_{k=1}^{\infty} N_k\right)}_{\text{null set}}$

$f(x,\cdot) = \lim_{k\to\infty} \varphi_k(x,\cdot)$ where φ_k is measurable,

it follows that $f(x,\cdot)$ is measurable on $\mathbb{R} \setminus \left(\bigcup_{k=1}^{\infty} N_k\right)$.

b) That $\int_{\mathbb{R}^2} f(x,y)\, dm_{\mathbb{R}^2} = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x,y)\, dm(y) \right] dm(x)$

Notice that for every $x \in \mathbb{R} \setminus \left(\bigcup_k N_k \right)$

$\ell_k(x,\cdot) \nearrow f(x,\cdot)$ so by the monotone convergence theorem

$$\underbrace{\int_{\mathbb{R}} \ell_k(x,y)\, dm(y)}_{h_k(x)} = \underbrace{\int_{\mathbb{R}} f(x,y)\, dm(y)}_{h(x)}$$

and thus

~~Monotone conv agai~~

$$\underbrace{\lim_{k\to\infty} \int_{\mathbb{R}} h_k(x)\, dm(x)}_{} = \int_{\mathbb{R}} h(x)\, dm(x) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x,y)\, dm(y) \right] dm(x)$$

$$= \lim \int_{\mathbb{R}^2} \ell_k(x,y)\, dm_{\mathbb{R}^2} = \int_{\mathbb{R}^2} f(x,y)\, dm_{\mathbb{R}^2}$$

Monotone conv thm again!

$\square$