SF3626, Analysis for PhD students, The construction of the Lebesgue measure.

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Contents

1 Lecture 2. Finding the right Assumptions.

At the end of lecture 1 we noticed that there are bounded and increasing sequences of Riemann integrable functions $f_j(x) \nearrow f(x)$ (point-wise) without $f(x)$ being Riemann integrable. That sequences that converge in rather good ways (increasing point-wise) whose limits are not integrable is a serious shortcoming of the Riemann's definition of the integral.

We also realized that the integral might be more versatile if we use the approximation

$$\sum_{j=-N}^{N} \epsilon j m\left(\{x;\ j\epsilon < f(x) \leq (j+1)\epsilon\}\right) \leq$$

$$\leq \int_\alpha^\beta f(x)dx \leq \sum_{j=-N}^{N} \epsilon(j+1) m\left(\{x;\ j\epsilon < f(x) \leq (j+1)\epsilon\}\right),$$

where $m(A)$ is some measure of the length of the set A.

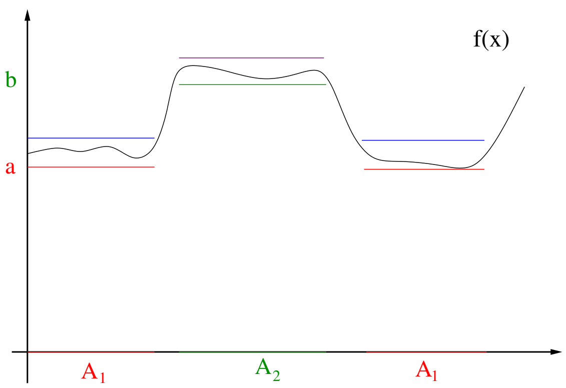

Figure: *The general idea with the Lebesgue integral is to approximate the area under the graph of f by functions that are constant where f is almost constant. In the figue we may approximate the area under the graph in on the red part of the $x-$axis by a function that has constant value a for all x in the red part of the x-axis denoted A_1, and by a constant function with value b on the green part A_2. In order to calculate the area we must find a good way to measure the length of the sets A_1 and A_2.*

The main problem is how are we supposed to define a reasonable measure of length? For simple sets, as intervals it is quite clear that the length of the interval (a, b) should be $b - a$. But the sets $\{x;\ j\epsilon < f(x) \leq (j+1)\epsilon\}$ may be very complicated and have huge oscillations; imagine the set

$$\left\{x \in (0,1);\ 10^{-3} < \sin(e^{-1/x}) \leq 2 \cdot 10^{-3}\right\} \tag{1}$$

which will certainly have infinitely many components - and we might come up with even worse examples[1] of sets whose length we would like to measure.

There are certain things that we would want our measure (of length) to satisfy. In particular, if we denote by $m(A)$ the measure of the set $A \subset \mathbb{R}$ then the following should hold

1. Any interval has its “natural length”: $m((a, b)) = b - a$ for any interval $(a, b) \subset \mathbb{R}$.
2. The measure should be countable additive: $m(\cup_j A_j) = \sum_{j=1}^{\infty} m(A_j)$ for any countable disjoint collection of sets A_j.

Later we will see that it is not possible to define any measure m, defined on all subsets of $\mathbb{R}$, in a way that satisfies these criteria.

A rather interesting fact is that open sets have length that is intuitively well defined because of the following lemma.

Lemma 1.1. *Let $U \subset \mathbb{R}$ be an open set. Then $U = \bigcup_{j=1}(a_j, b_j)$ where (a_j, b_j) are countable (or finite) and disjoint set of open intervals.*

Proof: Each connected component U_i of U is open and therefore contains a rational point $q_i \in \mathbb{Q}$, fix one such point $q_i \in \mathbb{Q}$ for each connected component of U. We may therefore define an injection from the connected components of U into a subset of $\mathbb{Q}$ that takes the connected component U_i to q_i. Therefore there is a bijection between the connected components of U and a subset of the countable set $\mathbb{Q}$. Therefore there are at most countable many connected subsets of U. Clearly each connected and open set in $\mathbb{R}$ is an interval. □

Since we would want the length of an interval, (a, b), to be $b - a$ it would be natural to define the length of an open set $U = \bigcup_{j=1}(a_j, b_j) \subset \mathbb{R}$ to be $\sum_j (b_j - a_j)$. Since there are at most countable many intervals the sum is well defined, though it might be diverge to ∞. Also since all terms $b_j - a_j > 0$ the summation is independent of the order of summation. We can therefore ascribe a length of an open set in a natural way. We will use this to define an outer measure (of length).

[1] The example in (1) is not bad at all from the point of view of integration. The function $\sin(e^{-1/x})$ is Riemann integrable on $(0, 1)$.

Definition 1.1. *We define the outer Lebesgue measure* m^* *on subsets* $A \subset \mathbb{R}$ *to* $[0, \infty)$ *according to*

$$m^*(A) = \inf \sum_{j=0}^{\infty} (b_j - a_j), \tag{2}$$

where the infimum is taken over all countable (or finite[2]*) unions of open intervals* (a_j, b_j) *such that* $A \subset \bigcup_{j=1}^{\infty}(a_j, b_j)$.

This definition makes perfect intuitive sense. However it is not absolutely clear that $m^*(a, b) = b - a$ with this definition. We need the following lemma.

Lemma 1.2. *The Lebesgue outer measure satisfies* $m^*([a, b]) = m^*((a, b)) = m^*([a, b)) = m^*((a, b]) = b - a$.

Proof: We begin by showing that $m^*([a, b]) = b - a$. The proof has several steps.

Step 1: *It is enough to consider finite coverings.*

Since $[a, b]$ is compact every open cover $U = \cup_{j=1}^{\infty}(a_j, b_j)$ reduces to a finite sub-cover $\tilde{U} = \cup_{j=1}^{N}(a_j, b_j)$. Clearly, since $b_j - a_j > 0$, $\sum_{j=1}^{N}(b_j - a_j) < \sum_{j=1}^{\infty}(b_j - a_j)$. It is therefore enough to consider finite coverings.

Step 2: *It is enough to consider coverings consisting of one interval* (a_1, b_1).

Assume that we have a covering $\tilde{U} = \cup_{j=1}^{N}(a_j, b_j)$. Then, since $a \in U$ there exists one interval, say (a_1, b_1), so that $a \in (a_1, b_1)$. If $[a, b] \subset (a_1, b_1)$ we already have one interval that covers $[a, b]$ so lets assume that $[a, b] \not\subset (a_1, b_1)$. This means that $a < b_1 < b$ and therefore $b \in [a, b]$. There must be another interval, say (a_2, b_2), such that $B_1 \in (a_2, b_2)$. But then $(a_1, b_2), (a_3, b_3), ..., (a_N, b_N)$ also cover of $[a, b]$, with only $N - 1$ intervals, and furthermore

$$(b_2 - a_1) + (b_3 - a_3) + ... + (b_N - a_N) < \sum_{j=1}^{N}(b_j - a_j).$$

We have therefore shown that for any cover U, containing at least two intervals, there is another cover with one less interval and smaller sum. We may conclude that the smallest cover can be achieved with one interval.

Step 3: $m^*([a, b]) = b - a$.

Since, for any $\epsilon > 0$, $(a - \epsilon/2, b + \epsilon/2)$ is a cover of $[a, b]$ it follows that $m^*([a, b]) \leq b - a + \epsilon$ and therefore

$$m^*([a, b]) \leq b - a$$

. Also any cover (c, d) of $[a, b]$ must have $c < a \leq b < d$ which implies that $m^*([a, b]) \geq b - a$. The statement follows.

In order to show that $m^*((a, b)) = m^*([a, b)) = m^*((a, b]) = b - a$ we notice that, since $(a, b) \subset (a, b] \subset [a, b]$ and $[a, b) \subset [a, b]$, each of $m^*((a, b)) \leq b - a$, $m^*((a, b]) \leq b - a$ and $m^*([a, b)) \leq b - a$ hold. We have to prove the reverse inequalities.

[2]In case the collection of intervals is finite then, naturally, the summation in (2) will be over the index set of the intervals and not to infinity.

To that end, let us assume that we con find a cover $U = \cup_{j=1}^{\infty}(a_j, b_j)$ of any of the intervals such that $\sum_{j=1}^{\infty}(b_j - a_j) < b - a$, say $\sum_{j=1}^{\infty}(b_j - a_j) = b - a - 5\delta$ for some $\delta > 0$. Then

$$(a-\delta, a+\delta)\bigcup(b-\delta, b+\delta)\bigcup\cup_{j=1}^{\infty}(a_j, b_j)$$

is a cover of $[a, b]$. This would imply that $m^*([a, b]) \leq b - a - \delta$ contradicting the first part of the proof. We may conclude that $m^*((a, b)) \geq b - a$, $m^*((a, b]) \geq b - a$ and $m^*([a, b)) \geq b - a$. This finishes the proof. □

The next simple lemma that we need is.

Lemma 1.3. *[Monotonicity of the measure.] If $A \subset B$ then $m^*(A) \leq m^*(B)$.*

proof: This follows from the fact that every open cover of B is also an open cover of A. Therefore

$$\inf_{A \subset U} \sum_{j=1}^{\infty}(b_j - a_j) \leq \inf_{B \subset U} \sum_{j=1}^{\infty}(b_j - a_j),$$

where the infinmum is taken over all the open sets $U = \cup_{j=1}^{\infty}(a_j, b_j)$. □

A somewhat refined estimate, the sub-additivity of the measure will be very important later. The main thing we would expect from a measure of length, besides that $m^*([a, b]) = b - a$, is that it is additive $m^*(\cup_j A_j) = \sum_j m^*(A_j)$ for disjoint sets A_j. When proving that m is additive we will repeatedly have to prove statements like $m(\cup_j A_j) = \sum_j m(A_j)$; that is $m(\cup A_j) \leq \sum m(A_j)$ and $m(\cup A_j) \geq \sum m(A_j)$. The following lemma proves one of the inequalities, and it will be referred to frequently in the next lecture.

Lemma 1.4. [Sub-additivity of the outer measure.] *Let A_j be a countable collection of sets in $\mathbb{R}$ then*

$$m^*\left(\cup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} m^*(A_j). \tag{3}$$

Proof: Using the definition of m^* to write (3) we get

$$\inf_{\cup_{j=1}^{\infty} A_j \subset U_{\cup}} \sum_{k=1}^{\infty}(d_k - c_k) \leq \sum_{J=1}^{\infty}\left(\inf_{A_j \subset U_j} \sum_{l=1}^{\infty}(b_{j,l} - a_{j,l})\right) = \inf_{A_j \subset U_j} \sum_{J=1}^{\infty}\sum_{l=1}^{\infty}(b_{j,l} - a_{j,l})$$

where $U_{\cup} = \cup_{k=1}^{\infty}(c_k, d_k)$ and $U_j = \cup_{l=1}^{\infty}(a_{j,l}, b_{j,l})$ and all the summations and unions are countable.

The lemma follows from noticing that the countable union of the countable collections of intervals $(a_{j,l}, b_{j,l})$ is still a countable collection that we may take as (c_k, d_k). So with any choice on the right side is also a choice on the left side. This yields the lemma. □

Lemma 1.5. *If A is the disjoint union of countably many intervals (open, closed or half open) with endpoints a_j and b_j, and A is bounded,*[3] *then*

$$m^*(A) = \sum_{j=1}^{\infty}(b_j - a_j).$$

[3]Bounded is not really needed, but it makes the proof simpler.

Sketch of the Proof: Let us first show the lemma for $A = \cup_{j=1}^{\infty}[a_j, b_j]$.

We claim that, for any finite N,

$$m^*(\cup_{j=1}^{N}[a_j - b_j]) = \sum_{j=1}^{N}(b_j - a_j). \tag{4}$$

Since $\cup_{j=1}^{N}[a_j - b_j]$ is compact it is enough to consider finite open coverings when calculating $m^*(\cup_{j=1}^{N}[a_j - b_j])$. But arguing as in Lemma 1.1 it is easy to see that each interval $[a_j, b_j]$ is covered by one open interval in the covering. Next one can easily show that if any open cover has a connected interval containing two adjacent intervals (adjacent is well defined since N is finite) then we may decrease the outer measure of the cover by splitting that interval into two. It follows that each $[a_j, b_j]$ there is no loss of generality in covering $\cup_{j=1}^{N}[a_j - b_j]$ by disjoint open intervals, each containing exactly one of the $[a_j, b_j]$. The equality (4) easily follows.

By monotonicity of the outer measure and (4):

$$\sum_{j=1}^{N}(b_j - a_j) = m^*(\cup_{j=1}^{N}[a_j - b_j]) \leq m^*(A).$$

Letting $N \to \infty$ implies that

$$\sum_{j=1}^{\infty}(b_j - a_j) \leq m^*(A) \leq \sum_{j=1}^{\infty}(b_j - a_j),$$

where the last inequality follows from sub-additivity of the measure.

In case some, or all, of the intervals I_j that define $A = \cup_{j=1}^{\infty} I_j$ are open or half open the lemma still holds. Let us briefly indicate why. By sub-additivity and Lemma 1.1 it follows that

$$m^*(A) \leq \sum_{j=1}^{\infty} m^*(I_j) = \sum_{j=1}^{\infty}(b_j - a_j),$$

it is therefore enough to show that

$$m^*(A) \geq \sum_{j=1}^{\infty}(b_j - a_j).$$

Arguing by contradiction we assume that, for some $\delta > 0$

$$m^*(A) = \sum_{j=1}^{\infty}(b_j - a_j) + \delta.$$

This means that there exists an open cover $\cup_{j=1}^{\infty}(c_j, d_j)$ of A such that

$$\sum_{j=1}^{\infty}(c_j - d_j) \leq \sum_{j=1}^{\infty}(b_j - a_j) + \frac{\delta}{2}.$$

If we adjoin the intervals

$$\left(a_j - \frac{\delta}{8 \cdot 2^{-j}}, a_j + \frac{\delta}{8 \cdot 2^{-j}}\right) \text{ and } \left(b_j - \frac{\delta}{8 \cdot 2^{-j}}, b_j + \frac{\delta}{8 \cdot 2^{-j}}\right)$$

to the collection (c_j, d_j) then we get an open cover $U = \cup_{j=1}^{\infty}(e_j, f_j)$ of $\cup_{j=1}^{\infty}\overline{I}_j$ such that

$$\sum_{j=1}^{\infty}(f_j - e_j) = \sum_{j=1}^{\infty}(d_j - c_j) + 2\sum_{j=1}^{\infty} 2\frac{\delta}{8 \cdot 2^{-j}} < \sum_{j=1}^{\infty}(b_j - a_j) = m^*(\cup_{j=1}^{\infty}\overline{I}_j),$$

where we used the first part of the argument in the last equality.[4] But since $\cup_{j=1}^{\infty}(e_j, f_j)$ is an open cover of $\cup_{j=1}^{\infty}\overline{I}_j$ we would get a contradiction. □

The Lemma 1.1 shows that the outer measure m^* at least behaves the way we want on intervals. The good thing with the definition of m^* is that it also gives a well defined length of all sets $A \subset \mathbb{R}$. However, and rather amazingly it turns out that there is no measure μ whatsoever that is defined on all sets $A \subset \mathbb{R}$ that has the good properties that we would expect of a measure.

Proposition 1.1. [VITALI SETS] *There is no non-negative function $\mu : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R}_+$, $\mu \neq 0$ and $\mu(A) \neq \infty$ for bounded sets A, such that:* [5]

1. *$\mu(A) = \mu(x_0 + A)$ for all sets $A \subset \mathbb{R}$.*[6]
2. *$\mu\left(\cup_{j=0}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$ for any countable and disjoint collection of sets A_j.*

Proof: Let us first show that the proposition holds on the set $\mathcal{P}([0,1))$ if we interpret all numbers modulo 1. At the end of the proof we will indicate how to treat the case stated in the proposition.

Define the equivalence relation $x \approx y$ if $x = y + q$ for some $q \in \mathbb{Q}$ (everything is calculated modulo 1). By using the axiom of choice we may form a set A consisting of one element from every equivalence class. Then each $x \in [0,1)$ may be written $x = y + q$ for $y \in A$ and $q \in \mathbb{Q} \cap [0,1)$. Since $q \in \mathbb{Q} \cap [0,1)$ is a countable set we may define the sets $A_j = q_j + A$, where $\{q_j;\ j \in \mathbb{N}\} = \mathbb{Q} \cap [0,1)$. Since every $x \in [0,1)$ can be written $x + q$ it follows that $\cup_{j=1}^{\infty} A_j = [0,1)$. By construction $A_j \cap A_k = \emptyset$ if $j \neq k$, this since A only contains one element form each equivalence class.

To summarize A_j forms a countable disjoint collection of sets such that

$$[0,1) = \cup_{j=0}^{\infty} A_j. \tag{5}$$

Furthermore, for each $j, k \in \mathbb{N}$,

$$A_j = A_k + q \qquad \text{for some } q \in \mathbb{Q} \cap [0,1). \tag{6}$$

Assume, aiming for a contradiction, that there a function μ as in the proposition then, in view of assumption 1 and (6), $\mu(A_j) = \mu(A_k)$ for all $j, k \in \mathbb{N}$. Also, from assumption 2 and (5), we may conclude that

$$\mu([0,1)) = \sum_{j=0}^{\infty} \mu(A_j). \tag{7}$$

[4]Here we are a little sketchy. In particular, even if I_j are disjoint it might not follow that $\overline{I}_j$ are disjoint. I am not quite sure that it is very interesting to investigate this here so I will leave it to the reader to clarify this point.

[5]We use the notation $\mathcal{P}(\mathbb{R})$ for all subsets of $\mathbb{R}$, or more generally $\mathcal{P}(S)$ is the set of all subsets of S. We also use the notation $\mathbb{R}_+ = \{x \in \mathbb{R};\ x \geq 0\}$.

[6]Here $x_0 + A$ is the translation of A by x_0; $x_0 + A = \{x_0 + x;\ x \in A\}$.

Since $\mu \neq 0$, and $\mu \geq 0$, it follows that $\mu([0,1)) > 0$ and therefor not all $\mu(A_j) \neq 0$. We conclude that $\mu(A_j) = c > 0$ for some j therefore, and because of (6), $\mu(A_j) = c > 0$ for all j. But if $\mu(A_j) = c > 0$ for all j then the series in (7) diverges which means that $\mu([0,1)) = \infty$ for a bounded set. We get a contradiction.

If we want to prove the same thing for $\mathbb{R}$ then we may argue similarly and define the set A as containing one representative from each equivalence class and then define

$$A_j = \underbrace{\{x + q_j;\ x \in A,\ x + q_j \in [0,1)\}}_{=A_j^+} \bigcup \underbrace{\{x + q_j - 1;\ x \in A,\ x + q_j \in [1,2)\}}_{=A_j^-}.$$

Then, for all j, $\mu(A_j) = \mu(A)$ since $\mu(A_j) = \mu(A_j^+ \cup A_j^-) = \mu(A_j^+) + \mu(A_j^-) = \mu(A_j^+) + \mu(1 + A_j^-) = \mu(q_j + A) = \mu(A)$. We arrive at

$$\mu([0,1)) = \sum_{j=1}^{\infty} \mu(A_j). \tag{8}$$

Since the series in (8) cannot be ∞ since the left side is the measure of a finite set we can conclude that $\mu(A_j) = 0$, and therefore $\mu([0,1)) = 0$. It follows that

$$\mu(\mathbb{R}) = \mu\left(\cup_{k\in\mathbb{Z}}[k, k+1)\right) = \sum_{k\in\mathbb{Z}} \mu([0,1)) = 0,$$

where we used translation invariance (assumption 1) in the last equality. But $\mu(\mathbb{R}) = 0$ is a contradiction to $\mu \neq 0$. $\square$

Since any reasonable definition of what length is should include the assumptions 1 and 2 we need to define the measure on a smaller domain than $\mathcal{P}(\mathbb{R})$. It is not absolutely clear what domain is the right domain of definition of the measure m^*. It turns out that the right definition of the (restricted) domain of m^* are the measurable sets.

Definition 1.2. *We define the Lebesgue measure m to be equal to m^* on the sets $S \subset \mathbb{R}$ that satisfies, for all sets $X \subset \mathbb{R}$,*

$$m^*(S) = m^*(X \cap S) + m^*(X \cap S^c). \tag{9}$$

We call the sets S that satisfy (9) measurable and the collection of all measurable sets will be denoted $\mathcal{M}$.

That is, the Lebesgue measure m is just m^* with domain of definition restricted to on the collection of measurable sets $\mathcal{M}$. In order for the measure to be useful we need to show that it satisfies some basic properties. In particular we would want the measure to satisfy the countable additivity condition

$$m(\cup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} m(A_j) \tag{10}$$

for each countable collection of measurable sets A_j. But for the countable additivity condition to be meaningful we need $\cup_{j=1}^{\infty} A_j$ to be measurable whenever the sets A_j are.

Defining the Lebesgue measure and measurable sets the way we do leads to two big questions. First: Will m satisfy the countable additivity condition (10)? Second: Which sets are measurable? If the class of measurable sets is to small then the Lebesgue measure will be useless.

We will show that all open sets are measurable and also that the measurable sets forms a $\sigma-$algebra - being a $\sigma-$algebra implies that the set of measurable sets $\mathcal{M}$ is rich and flexible enough to use for integration.

Definition 1.3. *Let R be a set and $\mathcal{S}$ be a collection of subsets of R. Then we say that $\mathcal{S}$ is a $\sigma-$algebra if*

1. $\emptyset \in \mathcal{S}$ *and* $R \in \mathcal{S}$,
2. *if* $A \in \mathcal{S}$ *then* $A^c \in \mathcal{S}$ *and*
3. *if* $A_j \in \mathcal{S}$, $j \in \mathbb{N}$, *then* $\cup_{j=1}^{\infty} A_j \in \mathcal{S}$.

Remark: *Notice that the third condition also implies that finite unions $\cup_{j=1}^{N} A_j \in \mathcal{S}$. This follows by choosing $A_j = \emptyset$ for $j > N$.*

One of our aims will be to show that $\mathcal{M}$ is a $\sigma-$algebra of subsets of $\mathbb{R}$. That $\emptyset \in \mathcal{M}$ and $\mathbb{R} \in \mathcal{M}$ is clear from the definition of $\mathcal{M}$. It also follows directly from the definition of $\mathcal{M}$ that if $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$.[7] We therefore only need to show that if $A_j \in \mathcal{M}$, $j \in \mathbb{N}$, then $\cup_{j=1}^{\infty} A_j \in \mathcal{M}$. The last condition is, of course, what we need in order to prove the countably additivity condition for m.

It is rather easy to show that if $\mathcal{S}$ is a $\sigma-$algebra then $\mathcal{S}$ contains many more sets than what is obvious from the definition.

Proposition 1.2. *Let $\mathcal{S}$ be a $\sigma-$algebra of subsets of R then*

1. *if* $A, B \in \mathcal{S}$ *then* $A \setminus B \in \mathcal{S}$ *and*
2. *if* $A_j \in \mathcal{S}$, $j \in \mathbb{N}$, *then* $\cap_{j=1}^{\infty} A_j \in \mathcal{S}$.

Remark: *That finite intersections $\cap_{j=1}^{N} A_j \in \mathcal{S}$ if $A_j \in \mathcal{S}$ follows from the final condition by choosing $A_j = R$ for $j > N$.*

Proof of Proposition 1.2: The first statement follows from the second since $A \setminus B = A \cap (B^c)$ and if $B \in \mathcal{S}$ then $B^c \in \mathcal{S}$.

To show the second statement we just notice that

$$\bigcap_{j=1}^{\infty} A_j = \left(\cup_{j=1}^{\infty} A_j^c \right)^c.$$

□

We will end this lecture by arguing that the, admittedly rather technical, definition of a measurable set gives a rather natural definition of length.

Example: *Later we will prove that open and closed sets are measurable, but for this example we will assume these facts.*

Assume that $A \subset [0, 1]$ and we want to find the measure of A, assume also the measure of open and closed sets is well defined. The measure of A must

[7] This is clear form (9) which is symmetric in A and A^c. In particular, substituting A^c for A (and using $(A^c)^c = A$) will not change (9).

be greater than (or equal to) $\sup_{K\subset A} m^*(K)$, *K closed, and less than (or equal to)* $\inf_{A\subset U} m^*(U) = m^*(A)$, *$U$ open.*[8] *The measure of A would then be well defined if*

$$\sup_{K\subset A} m^*(K) = \inf_{A\subset U} m^*(U) = m^*(A). \tag{11}$$

But if $[0,1]\setminus K$, for $K \subset A$, is an open set that contains $[0,1]\setminus A$ therefore

$$\sup_{K\subset A} m^*(K) = m^*([0,1]) - \inf_{([0,1]\setminus A)\subset U} m^*(U) = m^*([0,1]) - m^*([0,1]\setminus A) \tag{12}$$

- notice that because of Lemma 1.1 the definition of the length of open sets is rather uncomplicated and unambiguous which means that it is rather uncomplicated to define the length of closed sets.

Using (12) in (11) we get that the length of A is "well defined" only if

$$m^*([0,1]) - m^*([0,1]\setminus A) = m^*(A) \Rightarrow m^*([0,1]) = m^*([0,1]\cap A^c) + m^*([0,1]\cap A),$$

which is exactly the condition we get in the definition of measurable set with $X = [0,1]$. That allow X to be a general set instead of an interval is a matter of adjusting to the tradition of measure theory.[9] *The point is that the condition of measurable more or less states that we can ascribe a measure to a set A if the largest closed set contained in A has the same measure as the smallest open set containing A. The spirit of the definition of measurable is that the "length" of a measurable set can be sandwiched between two sets whose measure we konw.*

2 Lecture 3. Proving that the Assumptions are Right.

In this lecture we will prove that the measurable sets really form a $\sigma-$algebra and that the Lebesgue measure satisfies the countable additivity condition. We also need to show that the set of measurable sets is rich, in particular we will show that the measurable sets contains all open sets. The material is rather technical but, in its own way, very amazing.

Our first goal is to show that all open sets are measurable. The proof is rather long so we will begin with a lemma.

Lemma 2.1. *Assume that U is open and that $I = (a,b)$ then*

$$m^*(I) = m^*(U\cap I) + m^*(U\cap I^c).$$

Proof: By Lemma 1.4 it follows that

$$m^*(I) \le m^*(U\cap I) + m^*(U^c\cap I).$$

Therefore we only need to show that

$$m^*(I) \ge m^*(U\cap I) + m^*(U^c\cap I). \tag{13}$$

[8] There might be a slight mystery why we want K to be closed and U to be open, but let us accept that.

[9] And that in more abstract cases, for sets different than $\mathbb{R}$, there might not be anything as natural as an interval to use.

By Lemma 1.1 we may write $U = \cup_{j=1}^{\infty}(a_j, b_j)$. We may also assume that U is bounded since we intersect U by a the bounded set I in each occurrence in (13). The argument will be split up into several steps.

Step 1: *There exists, for every $\epsilon > 0$, an N such that*

$$m^*(U) \leq \sum_{j=1}^{N}(b_j - a_j) + \epsilon.$$

There is no loss of generality to assume that each $(a_j, b_j) \cap I \neq \emptyset$ and, upon relabeling, to assume that $a_j < b_j < a_{j+1} < b_{j+1}$. Define $U_\epsilon = \cup_{j=1}^{N}(a_j, b_j)$ and notice that the closed set U_ϵ^c is the union of

$$(-\infty, a_1], [b_1, a_2], [b_2, a_3], ..., [b_{N-1}, a_N], [b_N, \infty).$$

Proof of Step 1: This is clear since $\sum_{j=1}^{\infty}(b_j - a_j)$ is convergent, by the assumption that U is bounded. From Lemma 1.5 may conclude

$$m^*(U) = \sum_{j=1}^{\infty}(b_j - a_j) \leq \sum_{j=1}^{N}(b_j - a_j) + \epsilon.$$

The final parts of step 1 are just there for some book-keeping and should be clear. Throwing out the intervals (a_j, b_j) that does not intersect I should not effect anything and that the complement of U^c is closed and have the stated form is elementary.

We have four different cases to consider:

1. $a \in [a_1, b_1]$ and $b \in [a_n, b_N]$ or
2. $a \in [a_1, b_1]$ and $b \in [b_N, \infty)$ or
3. $a \in (-\infty, a_1]$ and $b \in [a_n, b_N]$ or
4. $a \in (-\infty, a_1]$ and $b \in [b_N, \infty)$.

All cases are handled in a very similar fashion so we will assume that we are in case 2 and leave the other cases to the reader.

Step 2: *The following equality holds*

$$m^*(U_\epsilon \cap I) = (b_1 - a) + \sum_{j=2}^{N}(b_j - a_j).$$

Proof of Step 2: Notice that we may write $U_\epsilon \cap I$ as a disjoint union of intervals:

$$U_\epsilon \cap I = (a, b_1) \cup (a_2, b_2) \cup (a_3, b_3) \cup ... \cup (a_N, b_N)$$

Step 2 follows from Lemma 1.5.

Step 3: *The following equality holds*

$$m^*(U_\epsilon^c \cap I) = \sum_{j=1}^{N-1}(a_{j+1} - b_j) + (b - b_N).$$

Proof of Step 3: Similar to step 2. We may write $U_\epsilon^c \cap I$ as a disjoint union of intervals:

$$U_\epsilon \cap I = (b_1, a_2) \cup (b_2, a_3) \cup (b_3, a_4) \cup ... \cup (b_N, b)$$

Step 3 follows from Lemma 1.5.

Step 4: *For every* $\epsilon > 0$ *the following holds*

$$m^*(U \cap I) + m^*(U^c \cap I) \leq m^*(I) + \epsilon,$$

in particular (13) holds. This finishes the proof.

Proof of Step 4: Since $U \setminus U_\epsilon = \cup_{j=N+1}^\infty (a_j, b_j)$ and $\sum_{j=N+1}^\infty (b_j - a_j) < \epsilon$ it follows that

$$m^*(U \cap I) \leq m^*(U_\epsilon \cap I) + m^*\left((\cup_{j=N+1}^\infty (a_j, b_j)) \cap I\right) \leq$$

$$\leq m^*(U_\epsilon \cap I) + m^*\left((\cup_{j=N+1}^\infty (a_j, b_j))\right) < m^*(U_\epsilon \cap I) + \epsilon, \tag{14}$$

where we used sub-additivity and that $(\cup_{j=N+1}^\infty (a_j, b_j)) \cap I \subset (\cup_{j=N+1}^\infty (a_j, b_j))$ (together with the monotonicity of the measure).

Also, by monotonicity of the measure and $U^c \subset U_\epsilon^c$,

$$m^*(U^c \cap I) \leq m^*(U_\epsilon^c \cap I). \tag{15}$$

From (14) and (15) we conclude that

$$m^*(U \cap I) + m^*(U^c \cap I) < m^*(U_\epsilon^c \cap I) + m^*(U_\epsilon \cap I) + \epsilon =$$

$$= (b_1 - a) + \sum_{j=2}^{N} (b_j - a_j) + \sum_{j=1}^{N-1} (a_{j+1} - b_j) + (b - b_N) + \epsilon = b - a + \epsilon,$$

where we used step 2 and 3 in the middle equality. □

Proposition 2.1. *Every open set* U *is measurable.*

Proof: We need to show that, for any $X \subset \mathbb{R}$,

$$m^*(X \cap U) + m^*(X \cap U^c) = m^*(X).$$

By sub-additivity it is enough to show that

$$m^*(X \cap U) + m^*(X \cap U^c) \leq m^*(X), \tag{16}$$

again we will show the last inequality with an arbitrary small ϵ error.

Let $\epsilon > 0$ and find a cover $\cup_{j=1}^\infty I_j$ of X, $I_j = (a_j, b_j)$, of X such that

$$\sum_{j=1}^{\infty} (b_j - a_j) < m^*(X) + \epsilon, \tag{17}$$

this we can always do by the definition of $m^*(X)$ as the infimum of all series such as the left side in (17).

By Lemma 1.1 we may also write $U = \cup_{j=1}^\infty J_j$, where $J_j = (c_j, d_j)$ are disjoint intervals.

Using that $X \subset \cup_j I_j$, monotonicity of the outer measure and sub-additivity of m^* we may calculate

$$m^*(U \cap X) \leq m^*\left(U \cap (\cup_j I_j)\right) \leq \sum_{j=1}^{\infty} m^*(U \cap I_j). \tag{18}$$

And similarly

$$m^*(U^c \cap X) \leq \sum_{j=1}^{\infty} m^*(U^c \cap I_j) \tag{19}$$

From (18) and (19) we may conclude that

$$m^*(U \cap X) + m^*(U^c \cap X) \leq \sum_{j=1}^{\infty} m^*(U \cap I_j) + \sum_{j=1}^{\infty} m^*(U^c \cap I_j) = \tag{20}$$

$$= \sum_{j=1}^{\infty} \left(m^*(U \cap I_j) + m^*(U^c \cap I_j)\right).$$

From Lemma 2.1 we may conclude that

$$\sum_{j=1}^{\infty} \left(m^*(U \cap I_j) + m^*(U^c \cap I_j)\right) = \sum_{j=1}^{\infty} m^*(I_j) = \sum_{j=1}^{\infty} (b_j - a_j) < m^*(X) + \epsilon, \tag{21}$$

where we also used (17) in the last inequality.

From (20) and (21) we can conclude that

$$m^*(U \cap X) + m^*(U^c \cap X) < m^*(X) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary this proves (16) and finishes the proof of the proposition. □

Corollary 2.1. *Every closed set is measurable.*

Proof: Since closed sets are complements of open sets and the definition of measurable is symmetric w.r.t. the complement this follows directly from Proposition 2.1. □

In order to get a rich enough class of measurable sets to show that the integral has good convergence properties it is not enough to show that all the open and all the closed sets are measurable. We will need to show that null sets are measurable as well. First we need to define the concept of null sets.

Definition 2.1. *We say that a set $A \subset \mathbb{R}$ is a null set if*

$$m^*(A) = 0.$$

Proposition 2.2. *All the null sets are measurable.*

Proof: Let A be a null set then, by the monotonicity of the outer measure, $m^*(X \cap A) \leq m^*(A) = 0$ and $m^*(X \cap A^c) \leq m^*(X)$. This clearly implies that

$$m^*(X \cap A) + m^*(X \cap A^c) \leq m^*(X).$$

The reverse inequality follows from sub-additivity. □

Next we will show that m^* is countably additive on measurable sets: if A_j, $j \in \mathbb{N}$, is a countable collection of disjoint and measurable sets. Then $m^*(\cup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} m^*(A_j)$. Again this requires some work. We begin by showing that m^* is finitely additive on disjoint and measurable sets.

Lemma 2.2. [Finite additivity] *Let A_j, $j = 1, 2, ..., N$, be a finite collection of disjoint measurable sets. Then*

$$m^*(\cup_{j=1}^N A_j) = \sum_{j=1}^N m^*(A_j).$$

Proof: By induction on N. We start with the base case, $N = 2$, then we define $X = A_1 \cup A_2$. By the definition of measurable, using that A_2 is measurable, it follows that

$$m^*(A_1 \cup A_2) = m^*(X) = m^*(\underbrace{X \cap A_2}_{=A_2}) + m^*(\underbrace{X \cap A_2^c}_{=A_1}) = m^*(A_2) + m^*(A_1),$$

the Lemma follows for $N = 2$.

Assume that the lemma holds for all measurable and disjoint collections A_j, $j = 1, 2, ..., N$, we want to show that for any collection A_j, $j = 1, 2, ..., N+1$, of disjoint and measurable sets

$$m^*(\cup_{j=1}^{N+1} A_j) = \sum_{j=1}^{N+1} m^*(A_j). \tag{22}$$

We argue as in the base case and define $X = \left(\cup_{j=1}^N A_j\right) \cup A_{N+1}$. Then, since A_{N+1} is measurable,

$$m^*\left(\left(\cup_{j=1}^N A_j\right) \cup A_{N+1}\right) = m^*(X) = m^*(\underbrace{X \cap A_{N+1}}_{=A_{N+1}}) + m^*(\underbrace{X \cap A_{N+1}^c}_{=\cup_{j=1}^N A_j}) =$$

$$= m^*(A_{N+1}) + m^*\left(\cup_{j=1}^N A_j\right) = \sum_{j=1}^{N+1} m^*(A_j),$$

where we used that $m^*\left(\cup_{j=1}^N A_j\right) = \sum_{j=1}^N m^*(A_j)$ by the induction hypothesis in the last equality. The lemma follows by induction. $\square$

We are now ready to prove countable additivity.

Proposition 2.3. [Countable Additivity.] *Let A_j, $j = 1, 2, 3, ...$, be a countable collection of disjoint measurable sets. Then*

$$m^*(\cup_{j=1}^\infty A_j) = \sum_{j=1}^\infty m^*(A_j).$$

Proof: Again by sub-additivity of the measure it is enough to prove that

$$\sum_{j=1}^\infty m^*(A_j) \leq m^*\left(\cup_{j=1}^\infty A_j\right). \tag{23}$$

By monotonicity of the measure it follows that

$$m^*\left(\cup_{j=1}^\infty A_j\right) \geq m^*\left(\cup_{j=1}^N A_j\right) = \sum_{j=1}^N m^*(A_j). \tag{24}$$

Passing to the limit $N \to \infty$ in (24) gives (23). $\square$

We also need to show that the measurable sets $\mathcal{M}$ forms a $\sigma-$algebra. By the definition of measurable sets, in particular by the symmetry in taking complements it is clear that $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$. Also that $\emptyset \in \mathcal{M}$ (since $\emptyset$ is a null set) and that $\mathbb{R} \in \mathcal{M}$ (since $\mathbb{R} = \emptyset^c$) is clear. We will however have to show that if $A_j \in \mathcal{M}$, $j \in \mathbb{N}$, then $\cup_{j=1}^{\infty} A_j \in \mathcal{M}$. This will require some work. We begin with a lemma.

Lemma 2.3. *If $A_j \in \mathcal{M}$, $j = 1, 2, ..., N$, then $\cup_{j=1}^{N} A_j \in \mathcal{M}$.*

Proof: We will prove this by induction on N. We begin with the base case $N = 2$.

To no ones surprise (I hope) it is, by sub-additivity, enough to show the inequality

$$m^*([A_1 \cup A_2] \cap X) + m^*([A_1 \cup A_2]^c \cap X) \leq m^*(X),$$

for any set $X \subset \mathbb{R}$.

In order to show this we use that A_1 is measurable and calculate

$$m^*(X) = m^*(X \cap A_1) + m^*(X \cap A_1^c) =$$

$$= m^*(X \cap A_1) + m^*([X \cap A_1^c] \cap A_2) + m^*([X \cap A_1^c] \cap A_2^c), \qquad (25)$$

where we used that A_2 is measurable in the last equality, which implies that

$$m^*(X \cap A_1^c) = m^*([X \cap A_1^c] \cap A_2) + m^*([X \cap A_1^c] \cap A_2^c).$$

In order to continue we use sub-additivity on the first two terms on the right in (25)

$$m^*(X \cap A_1) + m^*([X \cap A_1^c] \cap A_2) \geq$$

$$\geq m^*([X \cap A_1] \cup [(X \cap A_1^c) \cap A_2]) = m^*(X \cap [A_1 \cup A_2]), \qquad (26)$$

where we used the equality

$$[X \cap A_1] \cup [(X \cap A_1^c) \cap A_2] = X \cap [A_1 \cup A_2]$$

in the last step of the calculation. We also rewrite the last term in (25) as

$$m^*([X \cap A_1^c] \cap A_2^c) = m^*(X \cap [A_1 \cup A_2]^c). \qquad (27)$$

Using (26) and (27) in (25) we can conclude that for any set $X \in \mathbb{R}$

$$m^*(X) \geq m^*(X \cap [A_1 \cup A_2]) + m^*(X \cap [A_1 \cup A_2]^c),$$

which implies that $A_1 \cup A_2$ is measurable.

The induction step is easy. Given sets A_j, $j =, 2, ..., N+1$, and assuming that then lemma holds for all collections consisting of at most N sets we can conclude that

$$\cup_{j=1}^{N} A_j = \underbrace{\left(\cup_{j=1}^{N} A_j\right)}_{\in \mathcal{M}} \bigcup A_{N+1} \in \mathcal{M},$$

since the first union to the right is measurable by the induction hypothesis. $\square$

We can now prove that $\mathcal{M}$ is closed under countable unions.

Proposition 2.4. *Let $A_j \in \mathcal{M}$, $j = 1, 2, 3, ...$, then $A = \cup_{j=1}^{\infty} A_j \in \mathcal{M}$.*

Proof: As always we only need to show that

$$m^*(X) \geq m^*(X \cap A) + m^*(X \cap A^c).$$

The trick is to write A as a countable disjoint union (in order to use additivity of the measure). To that end we recursively define $T_1 = A_1$ and $T_n = A_n \setminus \cup_{j=1}^{n-1} T_j$. Then each T_j is measurable since A_n is and, by Lemma 2.3, $\cup_{j=1}^{n-1} T_j$ is. We also define $L_n = \cup_{j=1}^{n} T_j = \cup_{j=1}^{n} A_j$, which is also measurable by Lemma 2.3. Therefore

$$m^*(X) = m^*(X \cap L_n) + m^*(X \cap L_n^c) \geq m^*(X \cap L_n) + m^*(X \cap A^c), \qquad (28)$$

by the monotonicity of the measure since $X \cap A^c \subset X \cap L_n^c$. Using the definition of L_n in (28) we conclude that

$$m^*(X) \geq m^*\left(X \cap (\cup_{j=1}^{n} T_j)\right) + m^*(X \cap A^c). \qquad (29)$$

Since T_n is measurable we may calculate

$$m^*\left(X \cap (\cup_{j=1}^{n} T_j)\right) =$$

$$= m^*\Big(\underbrace{\left[X \cap (\cup_{j=1}^{n} T_j)\right] \cap T_n}_{=X \cap T_n}\Big) + m^*\Big(\underbrace{\left[X \cap (\cup_{j=1}^{n} T_j)\right] \cap T_n^c}_{=X \cap (\cup_{j=1}^{n-1} T_j)}\Big) =$$

$$= m^*\big(X \cap T_n\big) + m^*\big(X \cap (\cup_{j=1}^{n-1} T_j)\big) = \{\text{repeat}\} = \sum_{j=1}^{n} m^*(X \cap T_j),$$

where the first two equalities shows how to “move out ” T_n from the measure and the “repeat” just indicates that we do the same argument again to “move out” T_{n-1} and then T_{n-1} et.c. Using the last equality in (29) we can conclude that

$$m^*(X) \geq \sum_{j=1}^{n} m^*(X \cap T_j) + m^*(X \cap A^c). \qquad (30)$$

Letting $n \to \infty$ in (30) we get

$$m^*(X) \geq \sum_{j=1}^{\infty} m^*(X \cap T_j) + m^*(X \cap A^c). \qquad (31)$$

But $A = \cup_{j=1}^{\infty} A_j = \cup_{j=1}^{\infty} T_j$ and thus by sub-additivity

$$m^*(X \cap A) = m^*\left(X \cap \left(\cup_{j=1}^{\infty} T_j\right)\right) =$$

$$= m^*\left(\cup_{j=1}^{\infty} (X \cap T_j)\right) \leq \sum_{j=1}^{\infty} m^*(X \cap T_j). \qquad (32)$$

Inserting (32) in (31) we can conclude that

$$m^*(X) \geq m^*(X \cap A) + m^*(X \cap A^c).$$

The proposition follows. □

Let us formulate the main results we have proven this far in a theorem.

Theorem 2.1. *The Lebesgue measure m, that is the outer Lebesgue measure m^* restricted to the collection of measurable sets $\mathcal{M}$, is a non-negative countable additive function.*

Furthermore, the collection of measurable sets $\mathcal{M}$ forms a $\sigma-$algebra.

This is terribly abstract and at this point it might be difficult to see if it was worth it to go through many very technical proofs in order to derive a this theorem. It is hardly the kind of theorem whose statement makes its applications obvious. In the next lecture we will see that this theorem is exactly what we need in order to define a versatile integral that behaves well under limits. And it is only when proving the theorems for the integral we will be able to see that the theory developed this far is right.

Before we define the integral it will be good to gain a little better understanding of what a measurable set is and how general or bad it can be. We will also need a "continuity" result for measures. We will begin to show that any measurable set can be written as the intersection of countable many open sets and a null set.

Proposition 2.5. *Let $A \subset \mathbb{R}$ be any set, then there exists a set $B = \bigcap_{j=1}^{\infty} U_j$, where U_j are open sets and $A \subset B$, such that*

$$m^*(A) = m^*(B). \tag{33}$$

If A is measurable then

$$m^*(B \setminus A) = 0,$$

in particular any measurable set differs from the intersection of countably many open sets by a null set.

Proof: For a general set A it follows directly from the definition of the outer measure m^* that there exists an open set U_j such that $A \subset U_j$

$$m^*(U_j) \leq m^*(A) + \frac{1}{j}, \tag{34}$$

We may assume that

$$... \subset U_j \subset U_{j-1} \subset ... \subset U_1. \tag{35}$$

If not we may define $\tilde{U}_1 = U_1$ and inductively $\tilde{U}_j = U_j \cap \tilde{U}_{j-1}$, then $\tilde{U}_j \subset \tilde{U}_{j-1}$ will satisfy (35). The sets $\tilde{U}_j$ will also satisfy (34), by the monotonicity of the measure and from $A \subset \tilde{U}_j \subset U_j$.

By monotonicity of the measure, (35) and (34) it follows that $B = \cap_{j=1}^{\infty} U_j$ satisfies

$$m^*(A) \leq m^*(B) \leq m^*(U_j) \leq m^*(A) + \frac{1}{j}$$

for any $j \in \mathbb{N}$. This implies (33).

If A is measurable then it follows from the definition of measurable, $A \subset B$ and (33) that

$$m(A) = m(B) = m(B \cap A) + m(B \setminus A) = m(A) + m(B \setminus A),$$

subtracting $m(A)$ from both sides gives $m(B \setminus A) = 0$. □

The decomposition in the previous proposition is very powerful, and exact, but it is based on an possibly infinite intersection. At times it might be beneficial to have a finitary approximation of a measurable set. We provide such a finite approximation in the next proposition.

Proposition 2.6. *Assume that A is bounded and measurable. Then, for any $\epsilon > 0$, there exists a finite union of open intervals $U = \cup_{j=1}^{N}(a_j, b_j)$ such that*

$$m(A\Delta U) < \epsilon,$$

here Δ is denotes the symmetric difference $A\Delta U = (A \setminus U) \bigcup (U \setminus A)$.

Proof: If A is measurable then there exists an open set $\hat{U} = \cup_{j=1}^{\infty}(a_j, b_j)$ such that $A \subset \hat{U}$ and

$$m(\hat{U}) < m(A) + \frac{\epsilon}{2}.$$

Choose N large enough so that

$$m(\hat{U}) < \sum_{j=1}^{N}(b_j - a_j) + \frac{\epsilon}{2}$$

and define $U = \cup_{j=1}^{N}(a_j, b_j)$. Then using sub-additivity and monotonicity

$$m(A\Delta U) \leq m(\underbrace{A \setminus U}_{\subset \hat{U}\setminus U}) + m(\underbrace{U \setminus A}_{\subset \hat{U}\setminus A}) \leq m(\hat{U} \setminus U) + m(\hat{U} \setminus A) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

The final result of this lecture is

Proposition 2.7. [Continuity of the Measure] *If A_j is a countable collection of measurable sets then*

1. *if $A_j \subset A_{j+1}$ and $A = \cup_{j=1}^{\infty} A_j$ then*
$$m(A) = \lim_{j\to\infty} m(S_j). \tag{36}$$

2. *if $A_{j+1} \subset A_j$, $m(A_1) < \infty$, and $A = \cap_{j=1}^{\infty} A_j$ then*
$$m(A) = \lim_{j\to\infty} m(S_j). \tag{37}$$

Proof: We begin with the first statement and assume that $A_j \subset A_{j+1}$. We construct the disjoint sets $B_j = A_j \setminus A_{j-1}$ (taking $A_0 = \emptyset$) then B_j forms a disjoint measurable collection and $A = \cup_{k=1}^{\infty} B_k$ and $A_j = \cup_{k=1}^{j} B_k$.

By countable additivity for the measure we may calculate

$$\lim_{j\to\infty} m(A_j) = \lim_{j\to\infty} m(\cup_{k=1}^{j} B_k) = \lim_{j\to\infty} \sum_{k=1}^{j} m(B_k) =$$

$$= \sum_{k=1}^{\infty} m(B_k) = m(\cup_{k=1}^{\infty} B_k) = m(A).$$

This proves the first statement.

The second part follows from the first part. Notice that the sets $C_j = A_1 \setminus A_j$ forms an increasing sequence of measurable sets and $\cup_{j=1}^{\infty} C_j = A_1 \setminus A$. Using that all sets are measurable and (36) (with C_j in place of A_j) at the indicated place we can derive that

$$m(A_1) - m(A) = m(A_1 \setminus A) = \{(36)\} = \lim_{j\to\infty} m(C_j) =$$

$$= \lim_{j\to\infty} (m(A_1) - m(A_j)) = m(A_1) - \lim_{j\to\infty} m(A_j).$$

The statement (37) follows by canceling $m(A_1)$ and multiplying by -1. $\square$