- Recall the definitions: $\pi = \pi_1 \dots \pi_n \in \mathfrak{S}_n$, $S \subseteq [n-1]$,

$$D(\pi) = \{ i : \pi_i > \pi_{i+1} \} \subseteq [n-1]$$

$$\beta_n(S) = |\{ \pi \in \mathfrak{S}_n : D(\pi) = S \}|$$

$$\alpha_n(S) = |\{ \pi \in \mathfrak{S}_n : D(\pi) \subseteq S \}| = \sum_{T \subseteq S} \beta_n(T).$$

- We proved $\alpha_n(S) = \binom{n}{s_1, s_2 - s_1, s_3 - s_2, \dots, n - s_k}$

where $S = \{ s_1 < s_2 < \dots < s_k \}$.

- Hence by Inclusion-Exclusion

$$\beta_n(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_n(T)$$

- If $T = \{ s_{i_1} < s_{i_2} < \dots < s_{i_j} \}$, then (set $s_0 = 0$, $s_{k+1} = n$)

$$\alpha_n(T) = \frac{n!}{(s_{i_1} - s_0)! \, (s_{i_2} - s_{i_1})! \dots (s_{k+1} - s_{i_j})}$$

$$= n! \, f(0, i_1) f(i_1, i_2) \dots f(i_j, k+1),$$

where $f(i,j) = 1/(s_j - s_i)!$

$$\therefore \quad \beta_n(S) = n! \sum_{1 \le i_1 < i_2 < \dots < i_j \le k} (-1)^{k-j} f(0, i_1) f(i_1, i_2) \dots f(i_j, k+1)$$

Exercise: Let $f : [0,k+1]\times[0,k+1] \longrightarrow \mathbb{C}$ be s.t. $f(i,i)=1\ \forall i$ and $f(i,j)=0$ if $i>j$.

Then
$$\sum_{1\le i_1<i_2<\cdots<i_j\le k} (-1)^{k-j} f(0,i_1)f(i_1,i_2)\cdots f(i_j,k+1)$$
$$= \det\left[f(i,j+1)\right]_{i,j=0}^{k}$$

Corollary: $\beta_n(S) = n!\det\left[\frac{1}{(s_{j+1}-s_i)!}\right]_{i,j=0}^{k}$

where $s_0=0$, $s_{k+1}=n$.

Exercise: Prove $\beta_n(S) = \det\left[\binom{n-s_i}{s_{j+1}-s_i}\right]_{i,j=0}^{k}$

Example: $n=7$ $S=\{2,4\}$

$$\beta_n(S) = 7!\left|\begin{matrix} \frac{1}{(s_1-s_0)!} & \frac{1}{(s_2-s_0)!} & \frac{1}{(s_3-s_0)!} \\ \frac{1}{(s_1-s_1)!} & \frac{1}{(s_2-s_1)!} & \frac{1}{(s_3-s_1)!} \\ 0 & \frac{1}{(s_2-s_2)!} & \frac{1}{(s_3-s_2)!}\end{matrix}\right|$$

$$= 7!\left|\begin{matrix} \frac{1}{2!} & \frac{1}{4!} & \frac{1}{7!} \\ 1 & \frac{1}{2!} & \frac{1}{5!} \\ 0 & 1 & 3!\end{matrix}\right|$$

$$= -7!\left|\begin{matrix} \frac{1}{2!} & \frac{1}{7!} \\ 1 & 1/5! \end{matrix}\right| + 7!\cdot 3!\left|\begin{matrix} \frac{1}{2!} & \frac{1}{4!} \\ 1 & 1/2! \end{matrix}\right|$$

$$= 7!\cdot 3!\left(\frac{1}{4}-\frac{1}{6}\right) - \left(\frac{7\cdot 6}{2}-1\right) = \frac{7!}{2} - 20 = 2500$$

Permutations with restricted positions

- We may represent permutations as configurations in $[n] \times [n]$:

$\pi = 3241$

$$G(\pi) = \text{"graph of } \pi\text{"} = \{ (i, \pi(i)) : 1 \le i \le n \}$$

- Let $B \subseteq [n] \times [n]$ be a "<u>board</u>" and define

$$N_j = N_j(B) = |\{ \pi \in \mathfrak{S}_n : |B \cap G(\pi)| = j \}|$$

$r_k = r_k(B) =$ number of k-element subsets of B s.t. no two elements have a common coordinate

$=$ number of ways to place k non-attacking rooks on B.

- $r_B(x) = \text{"rook polynomial"} = \sum_k r_k x^k$

$N_B(x) = \text{" hit polynomial"} = \sum_j N_j x^j$

Theorem: $N_B(x) = \sum_{k=0}^{n} r_k (n-k)! \, (x-1)^k$

Proof: Suffices to prove for $x = m \in \{1, 2, 3, \dots\}$

Let $\mathcal{A}_m = \{(\pi, f) : \pi \in \mathfrak{S}_n$ and $f : G(\pi) \to [m+1]$ and $f(i,j) = 1$ if $(i,j) \notin B\}$

Compute $|\mathcal{A}_m|$ in two ways.

(1). If $|G(\pi) \cap B| = k$, then $(\pi, f) \in \mathcal{A}_m$ iff f is equal to one on $G(\pi) \setminus B$

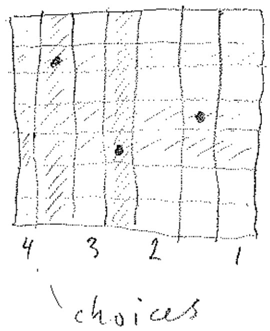
Hence there are $N_k \cdot (m+1)^k$ such f

$$\therefore \quad |\mathcal{A}_m| = \sum_k N_k (m+1)^k$$

(2). Fix a placement $A \subseteq B$ of non-attacking rooks with $|A| = k$.

How many $(\pi, f) \in \mathcal{A}_m$ have $f^{-1}([2, m+1]) = A$?

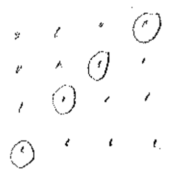
There are m^k choices for f on A, and $f \equiv 1$ outside A.



There are $(n-k)!$ ways to extend A to a permutation.

$$\therefore \quad |\mathcal{A}_m| = \sum_k r_k (n-k)! \, m^k$$

Example: $B = \{(i,i) : 1 \le i \le n\}$



$r_k = \binom{n}{k}$ so that

$$N_B(x) = \sum_{k=0}^{n} \binom{n}{k}(n-k)!\,(x-1)^k = \sum_{k=0}^{n} \frac{n!}{k!}(x-1)^k$$

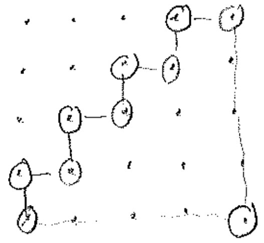
Note that $N_B(0) = N_0 = D(n)$ (derangements)

$$\therefore \quad D(n) = n!\sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Example: Manage problem:

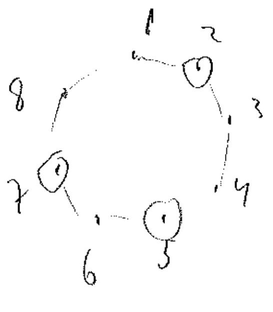
$M(n) = |\{\pi \in S_n : \pi(i) \neq i \bmod n,\ \pi(i) \neq i+1 \bmod n\}| = N_0(B)$, where

$$B = \{(1,1),(2,2),\dots,(n,n),(1,2),(2,3),\dots,(n-1,n),(n,1)\}$$



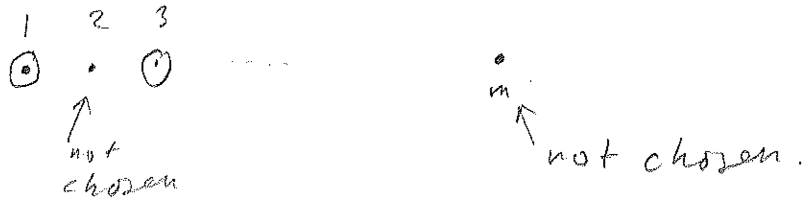
$r_k(B)$ = number of ways to choose k points from $2n$ points arranged on a circle so that no neighbours are chosen.

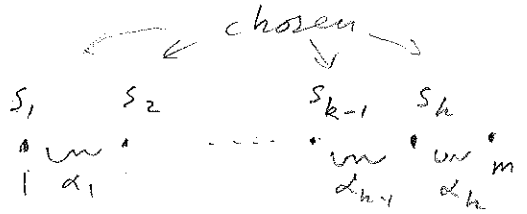
$=: R_{2n,k}$



<u>Lemma</u>: $R_{m,k} = \frac{m}{m-k}\binom{m-k}{k}$

<u>Proof</u>: (In how many ways $R'_{m,k}$ can we choose such a configuration so that 1 is chosen?





Let $\alpha_i = s_{i+1} - s_i - 1 \in \{1, 2, 3, \dots\}$

Then $\underset{\substack{\parallel \\ 1}}{s_1}, s_2, \dots, s_k$ is a configuration iff

$\alpha_1 + \alpha_2 + \dots + \alpha_k = m-k$ is a composition of $m-k$

in k parts.

$\therefore\ R'_{m,k} = \binom{m-k-1}{k-1}$

Let $\mathcal{O}_{m,k} = \{$configurations we seek$\}$. Then

$k|\mathcal{O}_{m,k}| = |\{(A,x) : x \in A\}| = m R'_{m,k}$,

since we may first choose $x \in [m]$ and then choose a configuration which contains x (in $R'_{m,k}$ ways).

$$R_{m,k} = \frac{m}{k}\binom{m-k-1}{k-1} = \frac{m}{m-k}\binom{m-k}{k}$$

Let $B = \{(1,1),(2,2),\dots,(n,n),(1,2),\dots,(n-1,n),(n,1)\}$

Corollary: N_B

$$N_B(x) = \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!\,(x-1)^k$$

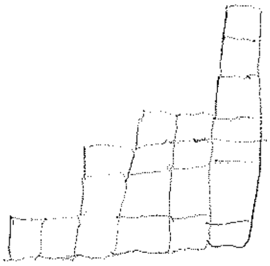
$$N_0(B) = \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!\,(-1)^k$$

Ferrers boards

Let $0 \le b_1 \le \dots \le b_m$ and

$$B = \{(i,j) : 1 \le i \le m \text{ and } 1 \le j \le b_i\} \qquad (*)$$

the Ferrers board defined by (b_i).

$B =$ 

$(b_1, b_2, b_3, b_4, b_5, b_6) = (1, 1, 3, 4, 4, 7)$

Theorem: Let B be as $(*)$ and set $s_i = b_i - i + 1$ for $1 \le i \le m$. Then

$$\sum_k r_k(B)\cdot(x)_{m-k} = \prod_{i=1}^{m}(x+s_i) \qquad (**)$$