F6

We have shown that the Lebesgue integral have good convergence properties. However, this is not enough for most applications. In particular, we want to have some sort of compactness for that we need to define a metric (we will define a norm) and have a space to work with.

There are several ways to define a space of integrable functions. The most common is

Definition: We ~~denote~~ define the space $L^p(D)$, where $D \subset \mathbb{R}^n$ is measurable, consisting of all equivalence classes of measurable functions u such that

$$\|u\|_{L^p(D)} = \left(\int_D |u(x)|^p \, dx \right)^{1/p} < \infty .$$

and $u \sim v$ if $\|u-v\|_{L^p(D)} = 0$

We need to show that $L^p(D)$ is a linear space and that $\|u\|_{L^p(D)}$ is a norm.

Proposition: (Hölder's inequality):
Assume that $u \in L^p(D)$, $v \in L^q(D)$, $\frac{1}{p} + \frac{1}{q} = 1$
Then $uv \in L^1$ and

~~Hold~~ $$\int_D u(x)v(x)\,dx \le \|u\|_{L^p(D)} \|v\|_{L^p(q)}$$

~~Proposition~~ 1

Proof: We can divide u and v by $\|u\|_{L^p}$ and $\|v\|_{L^q}$ respectively, then it is enough to show $\int_D u v \, dx \le 1$.

Remember Youngs inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \qquad \text{for all } a,b \ge 0 \quad \frac{1}{p}+\frac{1}{q}=1$$

Use it for $a = u(x)$, $b = v(x)$ and integrate

$$\int_D u(x)v(x) \le \int_D \frac{|u(x)|^p}{p}\,dx + \int_D \frac{|v(x)|^q}{q} = \frac{1}{p}\|u\|_{L^p}^p + \frac{1}{q}\|v\|_{L^q}^q = \frac{1}{p}+\frac{1}{q}$$

$\|u\|_{L^p} = \|v\|_{L^q} = 1.$

∎

Lemma: (Minkowski's ineq.) If $u, v \in L^p(D)$ then $u+v \in L^p(D)$ and the norm $\|\ \|_{L^p}$ satisfies the triangle inequality:

$$\|u+v\|_{L^p(D)} \le \|u\|_{L^p(D)} + \|v\|_{L^p(D)} . \qquad (1)$$

Proof:

That $u+v \in L^p$ follows from

$|u+v|^p \le 2^p|u| + 2^p|v|$.

To show (1) we calculate

$$\|f+g\|_{L^p}^p = \int_D |f+g|^p \le \int_D |f|\,|f+g|^{p-1}\,dx + \int_D |g|\,|f+g|^{p-1}\,dx \le$$

$$\le \{\text{Hölder}\} \le \left(\left(\int |f|^p\right)^{\frac{1}{p}} + \left(\int |g|^p\right)^{\frac{1}{p}}\right)\left(\int_D |f+g|^p\right)^{\frac{1}{q}} \Rightarrow \text{result.}$$ ∎

The most important property of an L^p-space is that it is complete.

Theorem: Every Cauchy sequence $u^j \in L^p(D)$, $1 \le p \le \infty$, ~~converges~~ converges to some element $u^0 \in L^p(D)$.
That is $L^p(D)$ is Cauchy complete.

~~We begin by proving the following Lemma, that is also important in probability theory~~

Strategy.

1) If $\forall \varepsilon > 0 \ \exists N > 0$ s.t.

$$n, m > N \implies \|u_n - u_m\|_{L^p(D)} < \varepsilon.$$

Then u_n and u_m cannot differ on a large set!

We should be able to find a sub-sequence u_{n_j} s.t. $\{x \mid |u_{n_j} - u_{n_l}| > \frac{\varepsilon}{2^k}\}$ has small measure if $j, l > N_k$

say $\underbrace{|\text{— " —}|}$ $< \frac{\varepsilon}{2^k}$.

2) Set — " — must differ for different j, l but since $\sum \frac{\varepsilon}{2^k}$ converges the different sets must still be small

So $u_{n_j} \to u_0$ a.e as $j \to \infty$.

3) Lebesgue spaces behave well under a.e. convergence (Fatou Lemma)
~~If~~ If a sub-sequence of a Cauchy sequence converges then the entire sequence converges.

Lemma: Let u_n be a Cauchy sequence in $L^p(D)$ then $\exists u_{n_j}$ s.t.

$$\|u_{n_j} - u_{n_l}\|_{L^p(D)} \leq \frac{1}{2^{2k}} \qquad \text{for } j, l \geq k.$$

Proof: Trivial.

$\boxtimes$

Lemma: Let u_{n_j} be the subsequence from the previous lemma. Then

$$\Big|\underbrace{\{x ;\ |u_{n_j} - u_{n_l}| \geq \tfrac{1}{2^k}\}}_{A}\Big| \leq \left(\frac{1}{2^k}\right)^p \qquad \text{for } j, l \geq k$$

Proof:

By the choice of the subsequence

$$\frac{1}{2^k} \cancel{\frac{1}{2^{2k}}} \geq \left(\int_D |u_{n_j} - u_{n_l}|^p dx\right)^{\frac{1}{p}} \geq \left(\int_A \underbrace{|u_{n_j} - u_{n_l}|^p}_{\geq 2^{-kp}} dx\right)^{\frac{1}{p}} \geq \cancel{2^{-k}} \underbrace{\left(\int_A dx\right)^{\frac{1}{p}}}_{= |A|^{\frac{1}{p}}}$$

$$\Rightarrow \quad |A| \leq \left(\frac{1}{2^k}\right)^p$$

$\boxtimes$

Lemma (Borel - Cantelli) Let A_k be a sequence of measurable sets and

$$\sum_{k=1}^{\infty} m(A_k) < \infty .$$

Then $A = \{ x \in \mathbb{R}^n ;\ x \in A_k$ for infinitely many k =

$$x \in \left[\bigcup_{k=m}^{\infty} A_k \right] \text{ for all } m \in \mathbb{N} \}$$

is a null set: $m(A) = 0$.

Proof: If $x \in A$ then $x \in \bigcup_{k=m}^{\infty} A_k = \bigcap_{n=1}^{m} \left(\bigcup_{k=n}^{\infty} A_k \right)$

for all m; $x \in \bigcap_{n=1}^{\infty} \left[\bigcup_{k=n}^{\infty} A_k \right] = A$

Thus

$$m(A) = m\left(\bigcap_{n=1}^{\infty} \left[\bigcap_{k=n}^{\infty} A_k \right] \right) = \lim_{n \to \infty} m\left(\bigcap_{k=n}^{\infty} A_k \right) \le \limsup_{n\to\infty} \sum_{k=n}^{\infty} m(A_k) = 0$$

Cont of measure

add of measure

∎

<u>Proposition:</u> Let u_{n_j} satisfy $\|u_{n_j}-u_{n_\ell}\|_{L^p(D)} \le \frac{1}{2^k}$ $\quad j,\ell \ge k$.

Then $u_{n_j} \to u_0 \in L^p$ pointwise and

$$\|u_{n_j}-u_0\|_{L^p(D)} \to 0. \qquad 1 \le p \le \infty$$

<u>Proof:</u> Define the sets (for simplicity of notation we write $u_{n_j} = u_j$ etc...)

$$A_k = \left\{ x \in D;\ |u_k - u_{k+1}| \ge \frac{1}{2^k} \right\}$$

Then $|A_k| \le \left(\frac{1}{2^k}\right)^p$ by previous Lemma

~~Since, by sub-additivity,~~

$$\cancel{m\left(\bigcup_k A_k\right) \le \sum_k m(A_k) \le \sum_{k=1}^{\infty} \left(\frac{1}{2^k}\right)^p \le \{p \ge 1\} \le \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.}$$

Also $\displaystyle \sum_{k=1}^{\infty} m(A_k) \le \sum_{k=1}^{\infty} \left(\frac{1}{2^k}\right)^p \le \{p \ge 1\} \le \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$

so by the Borel-Cantelli Lemma there exist a null set E s.t. for each $x \in D \setminus E$

x is only contained in finitely many of the A_k.

Pick such an x and say that $x \notin A_k$ for $k \ge k_0$

Then $\underbrace{}_{k,\ell \ge k_0}$ $\displaystyle |u_k(x) - u_\ell(x)| \le \sum_{m=k}^{\ell} \underbrace{|u_{m+1}(x) - u_m(x)|}_{\le \frac{1}{2^m} \text{ since } x \notin A_m} < \frac{1}{2^{k-1}}$

Thus $u_k(x)$ is Cauchy for $x \in D \setminus E$.

So $u_k(x) \to u_0(x)$ for a.e. x (every $x \in D \setminus E$)

Since u_0 is the limit of measurable functions it follows that u_0 is measurable.

Furthermore $|u_k - u_l| \geq 0$ is a sequence of integrable functions s.t

$$\left(\int_D |u_k-u_l|^p\right)^{1/p} \le \left\{\begin{matrix}\text{Minkowski} \\ \text{ineq}\end{matrix}\right\} \le \sum_{m=k}^{l-1} \|u_m - u_{m+1}\|_{L^p} \le \sum_{m=k}^{l-1}\left(\frac{1}{2^m}\right)$$

$$\Rightarrow \int_D |u_k-u_l|^p \le \left(\sum_{m=k}^{l-1}\frac{1}{2^m}\right)^p \le \left(\frac{1}{2^{k-1}}\right)^p$$

By Fatou's Lemma

$$\int_D \liminf_{l\to\infty} |u_k-u_l|^p \, dx = \int_D |u_k-u_0|^p \, dx \overbrace{\le}^{\text{Fatou's}} \liminf_{l\to\infty}\int_D |u_k-u_l|^p \le \frac{1}{2^{k-1}}$$

Thus

$$\lim_{k\to\infty}\int_D|u_k-u_0|^p \, dx \le \lim_{k\to\infty}\frac{1}{2^{k-1}} = 0.$$

$$\Rightarrow \quad \|u_k-u_0\|_{L^p} \to 0.$$

Theorem: $L^p(D)$ is Cauchy complete.

Proof: u_n Cauchy

Previous prop $\Rightarrow$ u_n convergent sub-sequence

Standard analysis $\Rightarrow$ the limit of a Cauchy sequence converges to the same limit as any convergent subsequence.

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