11.7 $X(n)$ WSS $1, 0, -1$ indep & equiprob. $E\{X(n)\} = 1\cdot\frac{1}{3} + 0\cdot\frac{1}{3} - 1\cdot\frac{1}{3} = 0$

$k \neq 0$ $E\{X(n)X(n+k)\} = E\{X(n)\}E\{X(n+k)\}$
$= 0$

$$Y(t) = \sum_n X(n)\,p(t - nT_S - \Theta)$$

Benet's :

$k=0$ $E\{X^2(n)\} = (1)^2\frac{1}{3} + 0^2\frac{1}{3} + (-1)^2(\frac{1}{3})$
$= 2/3$

$$S_Y(f) = \frac{1}{T_S}|P(f)|^2 S_X(fT_S)$$

$$S_Y(f) = \frac{1}{T_S}|P(f)|^2\cdot\frac{2}{3}$$

$$P(f) = \mathcal{F}\{Ce^{-at}u(t)\} = C\,\frac{1}{a+j2\pi f}$$

$$= \frac{2C^2}{3T_S}\left|\frac{1}{a+j2\pi f}\right|^2 = \frac{2C^2}{T_S 3(a^2+(2\pi f)^2)}$$

Then $$r_Y(\tau) = \mathcal{F}^{-1}\left(\frac{2C^2}{T_S3(a^2+(2\pi f)^2)}\right) = \mathcal{F}^{-1}\left(\frac{C^2}{3aT_S}\,\frac{2a}{a^2+(2\pi f)^2}\right) =$$

$$= \frac{C^2}{3aT_S}e^{-a|\tau|} \qquad E\{Y^2(t)\} = r_Y(0) = \frac{C^2}{3aT_S}e^{-a(0)} = \frac{C^2}{3aT_S}$$

RMS $$\sqrt{C^2/3aT_S} = C\sqrt{1/3aT_S} = 6\cdot\sqrt{1/3\cdot3} = \frac{6}{3} = \underline{2}\,V$$

12.6 $$R_X(f) = R_0 2^{-|f|/B}$$

$H(f)$ provides perfect reconstruction, hence $\hat{x}(t) = h(t) * x(t)$

$$E\{x^2(t)\} = 2R_0\int_0^\infty 2^{-f/B}df = 2R_0\int_0^\infty e^{-\ln(2)f/B} = 2R_0\left.\frac{e^{-\ln(2)f/B}}{-\ln(2)/B}\right]_0^\infty = 0 + \frac{2R_0B}{\ln(2)}$$

$$E\{(\hat{X}(t) - X(t))^2\} = E\{\hat{X}^2(t)\} + E\{X^2(t)\} - 2E\{X(t)\hat{X}(t)\} =$$

$$= E\{\hat{X}^2(t)\} + E\{X^2(t)\} - 2E\left\{X(t)\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau\right\} =$$

$$= E\{\hat{X}^2(t)\} + E\{X^2(t)\}$$
$$-2\int_{-\infty}^{\infty} h(\tau)E\{X(t)X(t-\tau)d\tau\} = E\{\hat{X}^2(t)\} + E\{X^2(t)\} - 2\int_{-\infty}^{\infty} h(\tau)r_X(\tau)d\tau$$

$$E\{\hat{X}^2(t)\} = \int_{-f_S/2}^{f_S/2} R_0 2^{-|f|/B} = 2R_0\int_0^{f_S/2} 2^{-|f|/B} = 2R_0\int_0^{f_S/2} e^{-\ln(2)f/B} =$$

$$= 2R_0\left.\frac{e^{-\ln(2)f/B}}{-\ln(2)/B}\right]_0^{f_S/2} = 2R_0\left(\frac{e^{-\ln(2)f_S/2B}}{-\ln(2)/B} + \frac{1}{\ln(2)/B}\right)$$

Parseval

$$\underbrace{\frac{2R_0B}{\ln(2)}\left(2 - e^{-\ln(2)f_S/2B}\right)}_{\triangleq G} - 2\int_{-\infty}^{\infty} h(\tau)r_X(\tau)d\tau = G - 2\int_{-\infty}^{\infty} H(f)S_X^*(f)df$$

$$= G - 2\int_{-f_S/2}^{f_S/2} S_X^*(f)df = E\{\hat{X}^2(t)\} = \frac{2R_0B}{\ln(2)}e^{-\ln(2)f_S/2B}$$

Hence the ratio : $\dfrac{\frac{2R_0B}{\ln(2)}}{\frac{2R_0B}{\ln(2)}e^{-\ln(2)f_s/2B}} = e^{\ln(2)f_s/2B} = 2^{f_s/2B}$

$10\log_{10}(2^{f_s/2B}) \geq 40 \quad f_s/2B \; 10\log_{10}(2) \geq 40 \quad f_s \geq \dfrac{8B}{\log_{10}(2)}$

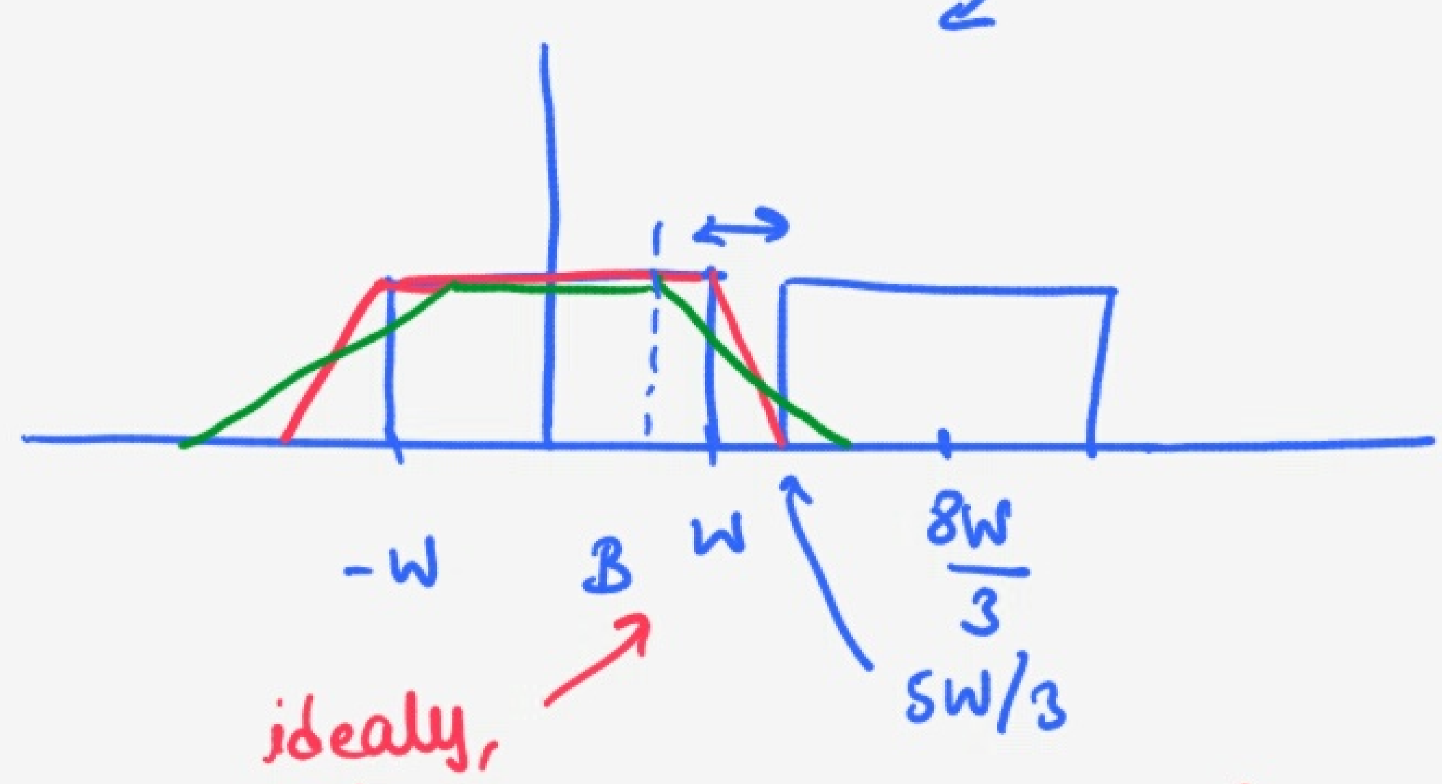
(12.18) Band-limited $x(t)$ with Fourier transform $X(f) = \begin{cases} 1 & |f|<W \\ 0 & \text{otherwise} \end{cases}$

$x(t)$ sampled $f_s = \dfrac{1}{T_s} = \dfrac{8W}{3}$ $\qquad \mathcal{E} = \displaystyle\int_{-\infty}^{\infty} |\hat{X}(f) - X(f)|\, df =$

$X_s(\nu) = \dfrac{1}{T_s} \sum_m' X\left(\dfrac{\nu - m}{T_s}\right)$

$\hat{X}(f) = \dfrac{1}{T_s} H(f) \sum_m' X\left(f - \dfrac{m}{T_s}\right) = \dfrac{1}{T_s} H(f) \underbrace{\sum_m X\left(f - \dfrac{8Wm}{3}\right)}_{2}$

$\displaystyle\int |\hat{X}(f) - X(f)|\, df =$



ideally, but

$5W/3 - W = 2W/3 < 4W/3$

we know that we have symmetry:

$2\displaystyle\int_0^{B+f_s/2} |\hat{X}(f) - X(f)|\, df =$

$= 2\displaystyle\int_0^{B} \underbrace{(\hat{X}(f) - X(f))}_{0}\, df + 2\int_B^{f_s/2} |\hat{X}(f) - X(f)|\, df =$

Therefore we will assume we have a suboptimality on both sides (the most general case)
*in case some of them is not needed it will appear from the equations.

= for $B \leq f \leq W$ error area of triangle upper corner

$\dfrac{5W}{3} \leq f \leq B + f_s/2$ error lower triangle

the line has slope (without sign) $2/f_s$

$\dfrac{h_1}{(W-B)} = \dfrac{2}{f_s} \qquad h_1 = \dfrac{2(W-B)}{f_s}$

$\dfrac{h_2}{\left(B + \frac{4W}{3} - \frac{5W}{3}\right)} = \dfrac{2}{f_s} \qquad h_2 = \dfrac{2(B - W/3)}{f_s}$

$2\left(\dfrac{(W-B)^2 2}{f_s} + \dfrac{2(B - W/3)^2}{f_s}\right) = \mathcal{E}$

$\dfrac{d\mathcal{E}}{dB} = \dfrac{-8}{f_s}(W-B) + \dfrac{8}{f_s}(B - W/3) = 0$

$2B - W - W/3 = 0 \quad 2B = \dfrac{4W}{3}$

$B = \dfrac{2W}{3} = \dfrac{4}{4}\,\dfrac{2W}{3} = \dfrac{1}{4}\,\dfrac{8W}{3} = \dfrac{1}{4} f_s.$