From: "A first course in chaotic dynamical systems"
by Robert Devaney

Example. Figure 11.6 shows a sketch of the graph of a piecewise linear* function defined on the interval $1 \leq x \leq 5$. Note that

$$F(1) = 3$$
$$F(3) = 4$$
$$F(4) = 2$$
$$F(2) = 5$$
$$F(5) = 1$$

so that we have a 5-cycle:

$$1 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 5 \mapsto 1 \ldots.$$

To see that F has no periodic point of period 3, we assume that there is such a point. From the graph, we see that

$$F([1,2]) = [3,5]$$
$$F([3,5]) = [1,4]$$
$$F([1,4]) = [2,5].$$

* Piecewise linear simply means that the graph of the function is a finite collection of straight lines.

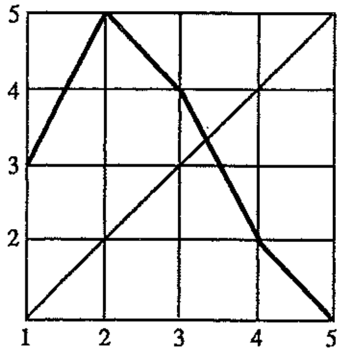

Fig. 11.6 F has period 5 but not period 3.

Hence, $F^3([1,2]) = [2,5]$ so $F^3([1,2]) \cap [1,2] = \{2\}$, which has period 5. Hence F^3 has no fixed points in $[1,2]$.

Similar arguments show that there are no 3-cycles in either of the intervals $[2,3]$ or $[4,5]$. We cannot use the same argument in the interval $[3,4]$, since F itself has a fixed point within this interval. However, we note that

$$F: [3,4] \to [2,4]$$

is a decreasing function. Also,

$$F: [2,4] \to [2,5]$$

and

$$F: [2,5] \to [1,5]$$

are also decreasing. The composition of an odd number of decreasing functions is decreasing. Hence

$$F^3: [3,4] \to [1,5]$$

is decreasing. Thus the graph of F^3 on $[3,4]$ meets the diagonal over $[3,4]$ in exactly one point. This point must be the fixed point of F. Therefore F has no 3-cycles in $[3,4]$ either. Consequently, this function has a period 5 point but no period 3 point.

Example. Figure 11.7 shows the graph of a piecewise linear function that has a 7-cycle given by

$$1 \mapsto 4 \mapsto 5 \mapsto 3 \mapsto 6 \mapsto 2 \mapsto 7 \mapsto 1 \mapsto \ldots .$$

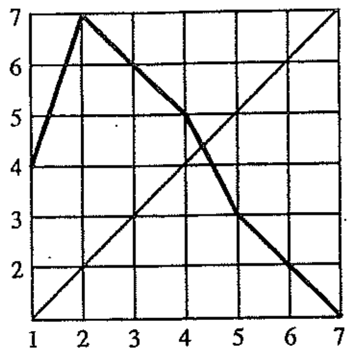

Fig. 11.7 F has a 7-cycle but no 5-cycle.

Arguments similar to those in the previous example show that F has no 5-cycle.

Fig. 11.15 A nonchaotic subshift of finite type.

Thus there are only two fixed points for this dynamical system, no other periodic points, and no dense orbit.

Exercises

1. Can a continuous function on **R** have a periodic point of period 48 and not one of period 56? Why?

2. Can a continuous function on **R** have a periodic point of period 176 but not one of period 96? Why?

3. Give an example of a function F: $[0,1] \to [0,1]$ that has a periodic point of period 3 and *no* other periods. Can this happen?

4. The graphs in Figure 11.16 each have a cycle of period 4 given by $\{0,1,2,3\}$. One of these functions has cycles of all other periods, and one has only periods 1, 2, and 4. Identify which function has each of these properties.

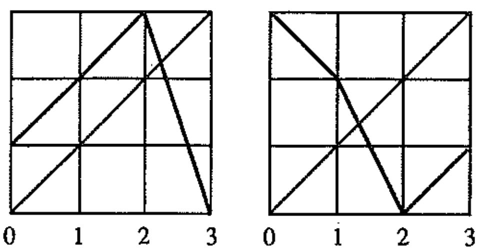

Fig. 11.16 Two graphs with period 4.

5. Suppose a continuous function F has a cycle of period $n \geq 3$ given by $a_1 < a_2 < \cdots < a_n$. Suppose that F permutes the a_i according to the rule $a_1 \mapsto a_2 \mapsto \cdots a_n \mapsto a_1$. What can you say about other cycles for F?

6. Consider the piecewise linear graph in Figure 11.7. Prove that this function has a cycle of period 7 but not period 5.