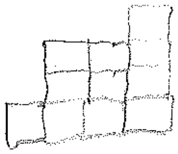
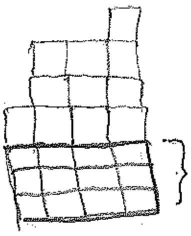
<u>Proof</u>: We prove (**) for any $x \in \mathbb{N}$.

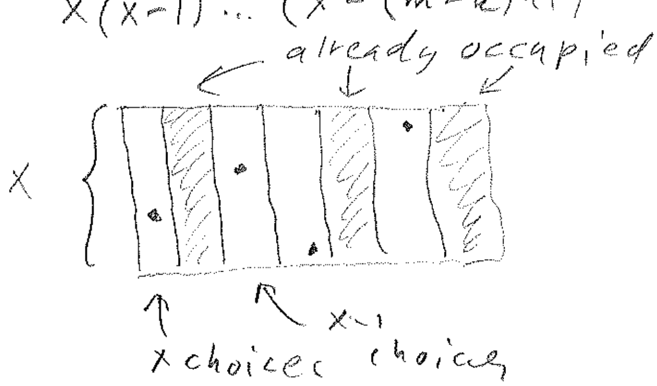
Let B_x be Ferrers board corresponding to

$(b_1+x, b_2+x, \dots, b_m+x)$

$B =$  , $B_3 =$  $= C_3$

Let us compute $r_m(B_x)$ in two ways:

(1). Place k rooks on B in r_k ways. Extend the placement to a rook placement on B_x in $x(x-1)\dots(x-(m-k)+1) = (x)_{m-k}$ ways



$$\therefore \quad r_m(B_x) = \sum_k r_k (x)_{m-k}$$

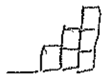
(2). Place a rook in first column of B_x in $x+b_1$ ways,

then — // — 2nd — // — $x+b_2-1$ ways,

— // — 3d — // — $x+b_3-2$

⋮

$$r_m(B_x) = \prod_{i=1}^{m} (x+b_i-i+1)$$

<u>Example:</u> $b_1, \dots, b_m = 0, 1, 2, \dots, m-1$ 

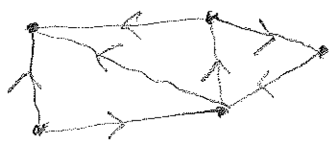
Then $\underline{s_i = 0}$ so that

$$x^m = \sum_k r_k \cdot (x)_{m-k} = \sum_k r_{m-k} \cdot (x)_k$$

Hence $r_{m-k} = S(m, m-k)$.

Determinants and nonintersecting paths

- A *directed graph* is a pair $\Gamma = (V, E)$ where V is a set of vertices and E is a set of *directed edges*. Each directed edge consists of two distinct vertices $x, y \in V$ and a direction $x \longrightarrow y$.



- A directed graph Γ is *acyclic* if there is no directed cycle in Γ:

$$x_1 \to x_2 \to x_3 \to \cdots \to x_k \to x_1$$

- A *source* in a digraph is a vertex x such that all edges connected to x points away from x.:

- A *sink* is a vertex y s.t. all edges connected to y points to y.

- We associate an indeterminate $w(i,j)$ to each directed edge $i \to j$.

- Let $s_1, \ldots, s_n$ be the sources and $z_1, \ldots, z_m$ –//– sinks

(NOTE! We actually don't require them to be sources and sinks)

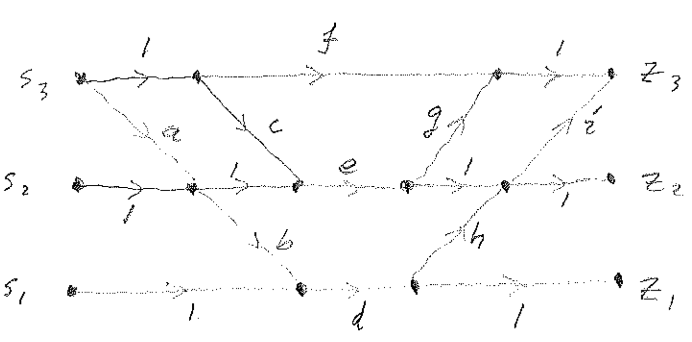
- If $p: x_1 \to x_2 \to \cdots \to x_k$ is a (directed) path in Γ, set $w(p) = w(x_1, x_2) \cdot w(x_2, x_3) \cdots w(x_{k-1}, x_k)$

- If $(p_1, \ldots, p_\ell)$ is a tuple of paths we set

$$\underset{p}{\overset{\shortparallel}{(p_1,\ldots,p_\ell)}} \qquad w(p) = w(p_1) \cdots w(p_\ell)$$

- Define an $n \times m$ matrix A by setting $A = (A_{ij})_{ij \in [n] \times [m]}$

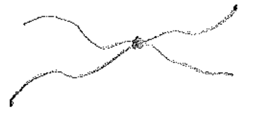
A_{ij} = sum of weights of paths between s_i and z_j.



$$A = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & d & dh & dhi \\ 2 & bd & bdh+e & bdhi+eg+ei \\ 3 & abd & abdh+ae+ce & f+(a+c)e(g+i)+abdhi \end{array}$$

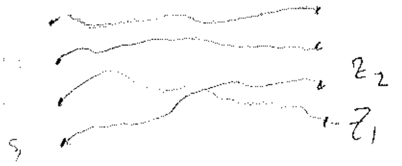
- If $I \subseteq [n]$ and $J \subseteq [m]$, $|I| = |J|$, let $A(I,J)$ be the determinant of the square matrix whose rows are indexed by I and columns by J (minor).

- A tuple of path $(P_1, \dots, P_l)$ is *intersecting* if there is a pair $1 \le i < j \le l$ such that P_i and P_j have a common vertex.



- Let $I \subseteq [n]$ and $J \subseteq [m]$, where $|I| = |J| = l$.
  $I = \{i_1 < i_2 < \dots < i_l\}$, $J = \{j_1 < j_2 < \dots < j_l\}$

- We consider tuples of paths $(P_1, \dots, P_l)$, where P_i is a path from s_i to $z_{\pi(i)}$ for all $1 \le i \le l$, where π is a permutation in $\mathfrak{S}_l$.



- I, J is <u>compatible</u> if $(P_1, \ldots, P_l)$ is intersecting whenever $\pi \neq$ identity.

<u>Theorem</u> ("Lindström's lemma").
Suppose I, J is compatible. Then

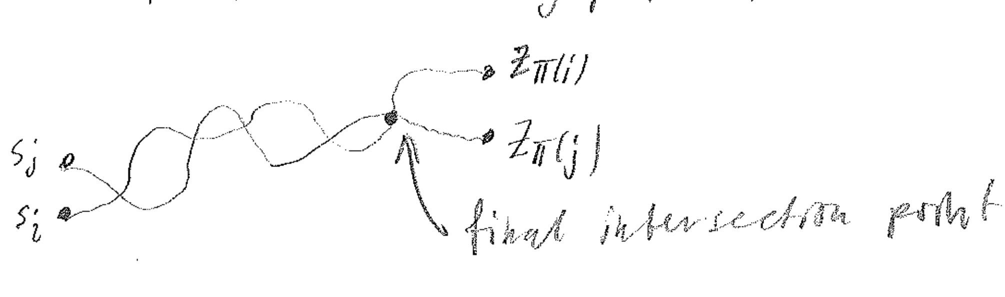
$$A(I,J) = \sum_{(P_1,\ldots,P_l)} w(P_1) \cdots w(P_l), \qquad (|I| = |J| = l)$$

where the sum is over all non-intersecting tuples of paths, where P_j is a path from s_{i_j} to z_{i_j} for all j.

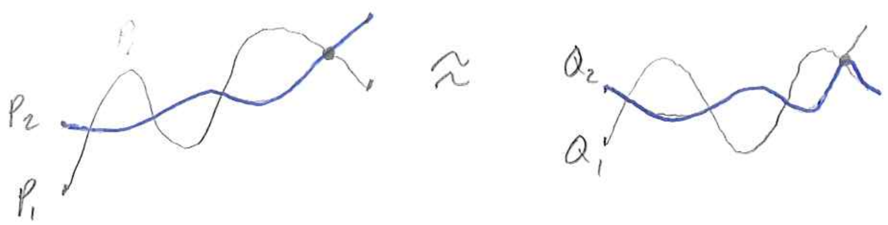
<u>Proof</u>: May assume $I = J = [n] = [m]$.
Note that $\det(A) = \sum_{\pi \in \mathfrak{S}_n} \operatorname{sign}(\pi) \prod_{i=1}^{n} A_{i\pi(i)}$

$$= \sum_{\substack{\pi \in \mathfrak{S}_n \\ \mathcal{P}_\pi = (P_1, \ldots, P_n)}} \operatorname{sign}(\pi)\, w(\mathcal{P}),$$

where $\mathcal{P}_\pi = (P_1, \ldots, P_n)$ ranges over all n-tuples of paths where P_i is a path from s_i to $z_{\pi(i)}$.
Consider the set, T, of all pairs of intersecting paths with different endpoints. For each such pair there is a final intersecting point:



Define an equivalence relation on T by saying that two pairs $\{P_1, P_2\}$ and $\{Q_1, Q_2\}$ are equivalent if $\{P_1, P_2\}$ is obtained from $\{Q_1, Q_2\}$ by "switching the ends" after the final intersection point.



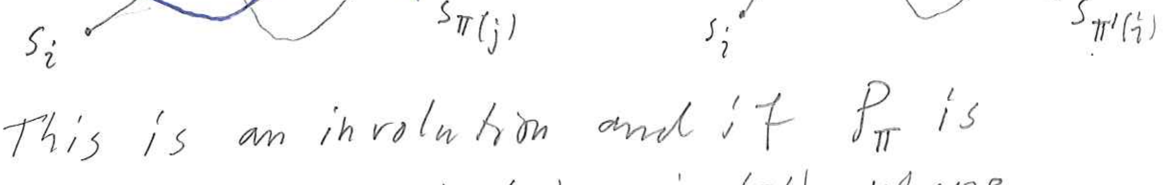
Hence there are exactly two members of each equivalence class.

Fix a total order on the equivalence classes.

Define an involution φ on the set of all tuples $\{P_\pi = (P_1, \dots, P_n) : \pi \in \mathfrak{S}_n\}$ appearing in the determinant:

If P_π is non-intersecting (and hence $\pi = \text{identity}$), then $\varphi(P_\pi) = P_\pi$.

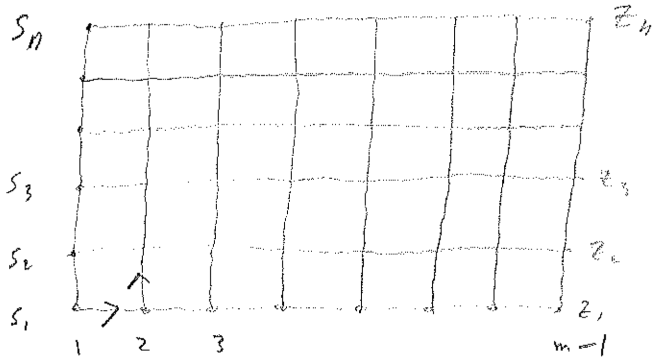
If $P_\pi = (P_1, \dots, P_n)$ is intersecting, then take the pair $\{P_i, P_j\}$ which is maximal with respect to the order on T, and switch the ends!



This is an involution and if P_π is intersecting, then $\mathrm{sign}(\pi) = -\mathrm{sign}(\pi')$ where $\varphi(P_\pi) = P_{\pi'}$.

Hence all intersecting tuples are cancelled from the determinant and we are left with the sum over all non-intersecting tuples. $\square$

<u>Ex</u>



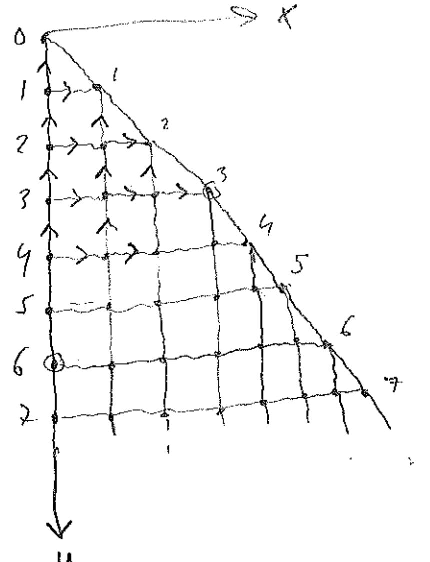
weight $= 1$

The number of paths between s_i and z_j is

$\binom{j-i+m}{j-i}$, since we should choose which $j-i$ of the $j-i+m$ steps should be vertical.

$$A = \begin{bmatrix} 1 & \binom{m+1}{1} & \binom{m+2}{2} & \binom{m+3}{3} & \cdots \\ 0 & 1 & \binom{m+1}{1} & \binom{m+2}{2} & \binom{m+3}{3} \cdots \\ 0 & 0 & 1 & \binom{m+1}{1} & \binom{m+2}{2} \cdots \\ & & \vdots & & \end{bmatrix}$$

Note that all I, J for which $|I| = |J|$ are compatible. Hence <u>all</u> minors of A are non-negative.



$$A_{ij} = \binom{i}{j}$$

$$A = \left[\binom{i}{j}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & & \\ 1 & 1 & 0 & 0 & 0 & & \\ 1 & 2 & 1 & 0 & 0 & & \\ 1 & 3 & 3 & 1 & 0 & & \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \end{bmatrix}$$

- Each minor of A is nonnegative
- Such matrices are called totally nonnegative.