# Posets

**Def.** A *partially ordered set*, or *poset*, is a pair $(P, \leq)$, where

- $P$ is a set,
- $\leq$ is a relation on $P$ satisfying:

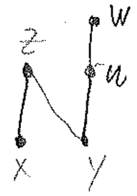
(a). Reflexivity: $x \leq x, \forall x \in P$

(b). Antisymmetry: $x \leq y \wedge y \leq x \Rightarrow x = y$

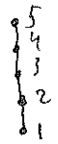
(c). Transitivity: $x \leq y \wedge y \leq z \Rightarrow x \leq z$

- We write $x < y$ if $x \leq y \wedge x \neq y$.
- $y$ is *covered* by $x$, written $x \lessdot y$, if $x < y$ and there is no element $z \in P$ s.t. $x < z < y$.
- We may represent a finite poset $P$ by its Hasse diagram: A graph with vertices $P$ and an edge "going up" between each $x, y$ with $x \lessdot y$:

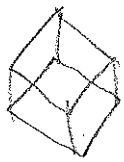
$$P = \left( \{x, y, z, u, w\}, \begin{array}{l} x \leq x,\ y \leq y,\ z \leq z,\ u \leq u,\ w \leq w \\ x \leq z,\ y \leq z,\ y \leq u,\ y \leq w,\ u \leq w, \\ y \leq w \end{array} \right)$$



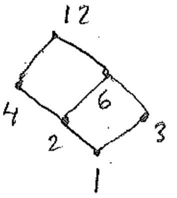

- Examples: • $\bar{n} = ([n], \leq)$. $\bar{n}$ is a *chain* or *total order*.



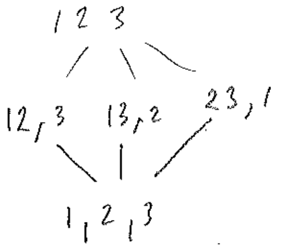

- $n \in \mathbb{N}$, $B_n = (2^{[n]}, \leq)$, where $A \leq B$ iff $A \subseteq B$ (Boolean algebra)

$B_1 =$  , $B_2 =$  , $B_3 =$ 

- If $n > 0$, let $D_n$ be the set of all <u>divisors</u> ($\geq 1$) of $n$ with order $a | b$ if $a$ divides $b$

$D_{12} =$ 


- Let $\Pi_n$ be the set of all partitions of $[n]$. $\pi \leq \sigma$ if each block of $\pi$ is contained in a block of $\sigma$.

$\Pi_3 =$ 


restricted to $X$

- Note that if $X \subseteq P$, then $(X, \leq)$ is a poset, a (induced) <u>sub-poset</u>.
- A <u>chain</u> in $P$ is a sub-poset of $P$ in which every pair of elements are comparable.
- A chain is <u>maximal</u> in $P$ if there is no proper chain containing it.
- The <u>length</u> of a chain $C$ is $l(C) = |C| - 1$.
- A poset $P$ is <u>graded</u> if every maximal chain has the same length $r$. $r$ is called the rank.
- 

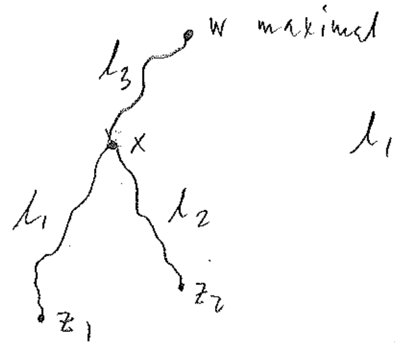
graded poset of rank 2

- If $x, y \in P$, with $x \leq y$; let
  $[x,y] = \{z \in P : x \leq z \leq y\}$ (interval)

- A *minimal* element in $P$ is an element $x \in P$ s.t. there is no $y \in P$ s.t. $y < x$.
  (maximal element defined similarly)

- If $P$ is graded, let $\rho : P \to [r]$ be defined by
  $\rho(x) =$ length of any maximal chain in $[z, x]$, where $z$ is a minimal element with $z \leq x$.

- The *rank function* $\rho$ is well-defined:




$$l_1 + l_3 = l_2 + l_3 \Rightarrow l_1 = l_2$$

- The *rank-generating function* of $P$ is
  $F(P,t) = \sum_{i=0}^{r} p_i t^i$, where
  $p_i$ is the number of elements of rank $i$.

*Example:*

| | Poset | $\rho(x)$ | $F(P,t)$ |
|---|---|---|---|
| | $\bar{n}$ | $x-1$ | $1+t+\dots+t^{n-1} = (n)_t$ |
| | $B_n$ | $\lvert X \rvert$ | $(1+t)^n$ |
| $x = p_1^{\beta_1} \dots p_k^{\beta_k}$, $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ prime-factorizations | $D_n$ | $\beta_1 + \dots + \beta_k$ | $(\alpha_1+1)_t (\alpha_2+1)_t \dots (\alpha_k+1)_t$ |
| | $\Pi_n$ | $n - \#$blocks of $x$ | $\sum_i S(n, n-i) t^i$ |

## Lattices

- $z$ is an upper bound of $x, y$ if $x \le z$, $y \le z$
- $z$ -//- lower bound -//- $z \le x$, $z \le y$
- $z$ is the least upper bound if $z \le w$ for all other u.b. Denoted $z = x \vee y$. ("$x$ join $y$")
- $z$ is the greatest lower bound if $z \ge w$ for all other l.b. Denoted $z = x \wedge y$. ("$x$ meet $y$")

**Def.** A lattice is a poset for which $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$.

Properties:
- $x \vee x = x \wedge x = x \quad \forall x$
- $x \wedge (x \vee y) = x = x \vee (x \wedge y) \quad \forall x, y$
- $x \wedge y = x \Leftrightarrow x \vee y = y \Leftrightarrow x \le y$
- commutative, associative

- Note that a finite lattice has a least element denoted $\hat{0}$ and greatest element denoted $\hat{1}$:

$$\hat{0} = \bigwedge_{x \in L} x \quad , \quad \hat{1} = \bigvee_{x \in L} x$$

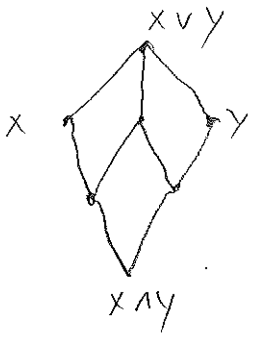
- Meet-semilattice: Every pair of elements has a meet.
- Join-semilattice: -//- join.

Proposition: Let $P$ be a finite meet-semilattice with $\hat{1}$. Then $P$ is a lattice.

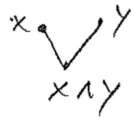
Proof: $x \vee y = \bigwedge_{\substack{z \ge x \\ z \ge y}} z$ □

Examples of lattices: $B_n, D_n, \Pi_n, \Pi$

- A finite lattice is <u>modular</u> if it is graded and $\rho(x)+\rho(y)=\rho(x\wedge y)+\rho(x\vee y)$ for all $x,y\in L$.
- For example $B_n$ (upper-)
- A finite lattice is (upper) <u>semimodular</u> if it is graded and $\rho(x)+\rho(y)\geq\rho(x\vee y)+\rho(x\wedge y)$ for all $x,y\in L$
- Example of semimodular but not modular




<u>Prop.</u> A finite lattice is semimodular iff
(*) If $x$ and $y$ cover $x\wedge y$, then $x\vee y$ covers $x$ and $y$.


<u>Proof:</u> Suppose $L$ is semimodular and



Then $\rho(x\vee y)\leq\rho(x)+\rho(y)-\rho(x\wedge y)=\rho(x)+1$

Hence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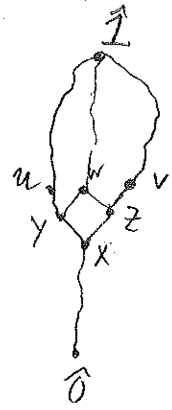
Assume (*) and prove that all maximal chains have the same length by induction on $|L|$.

Consider two max chains:




Let $x$ be the last intersection point (before $\hat 1$)

Then by induction the chains

$z < v < \dots < \hat 1$ since $|[z, \hat 1]| < |L|$

$z < w < \dots < \hat 1$

$z < u < \dots < \hat 1$

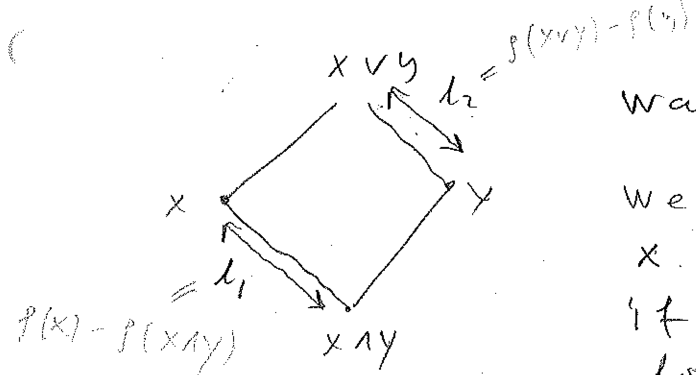
have the same length which proves that $L$ is graded.

Assume (*)

Let us prove $\rho(x) + \rho(y) \geq \rho(x \vee y) + \rho(x \wedge y)$

by induction over $|[x \wedge y, x \vee y]| = n$

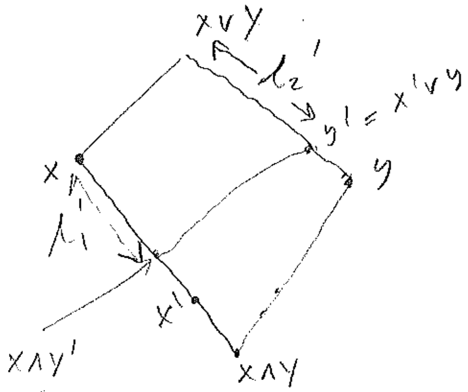
It is true for $n \leq 2$ (why?).




Want $l_1 \geq l_2$!

We may assume that $x$ does not cover $x \wedge y$, since if $x$ and $y$ cover $x \wedge y$, we are done by (*).

Let $x'$ be such that $x \wedge y \lessdot x' < x$ and let $y' = x' \vee y$.




Note that $x' \vee y \neq y$ (since otherwise $x' \leq y$ and thus $x' = x \wedge y$)

Hence $y \wedge x' = x \wedge y$.

By induction:

$1 = \rho(x') - \rho(x' \wedge y') \geq \rho(y') - \rho(y)$

Thus $\rho(y') = \rho(y) + 1$.

By induction and construction:

$l_1 - 1 \geq l_1' \geq l_2' = l_2 - 1$ □

- An *atom* of a poset with $\hat{0}$ is an element $x$ which covers $\hat{0}$.
- A finite *geometric* lattice is a finite semimodular lattice which is also atomic, i.e., each element $x \in L$ is a join of atoms
- Let $V$ be a vectorspace over some field $K$ and let $S = \{v_1, \dots, v_n\} \subseteq V$. All subsets of the form $A = S \cap W$ where $W \subseteq V$ is a subspace forms a lattice $L(S)$, where $A \leq B$ if $A \subseteq B$.

  Let $\langle A \rangle$ denote the subspace spanned by $A \subseteq S$. Then $A \in L(S) \Leftrightarrow S \cap \langle A \rangle = A$.

  Note that $S = \hat{1} = S \cap V$ and if $A = S \cap W_1$, $B = S \cap W_2$ then $A \cap B = S \cap (W_1 \cap W_2)$. Hence $A \wedge B = A \cap B$, so that $L(S)$ is a meet-semi lattice and hence a lattice. Also

  $$A \vee B = S \cap (\langle A \cup B \rangle).$$

  Hence $A = \bigvee_{a \in A} \{a\}$, and thus $L(S)$ is atomic.
- Note that $L(S)$ is graded and

  $$\rho(A) = \dim \langle A \rangle$$
- By the dimension-formula we have

  $$\underbrace{\dim \langle A \rangle}_{=\rho(A)} + \underbrace{\dim \langle B \rangle}_{=\rho(B)} = \dim(\langle A \rangle \cap \langle B \rangle) + \underbrace{\dim(\langle A \cup B \rangle)}_{=\rho(A \vee B)}$$
- Since $\rho(A \wedge B) = \rho(A \cap B) = \dim(\langle A \cap B \rangle)$
  $\leq \dim(\langle A \rangle \cap \langle B \rangle)$ (since $\langle A \cap B \rangle \subseteq \langle A \rangle \cap \langle B \rangle$)

$$\therefore \quad \rho(A) + \rho(B) \geq \rho(A \wedge B) + \rho(A \vee B)$$

and thus $L(S)$ is geometric.

Example :