KTH Teknikvetenskap

SF1624 Algebra och geometri
Exam
Friday, 21 October 2016

Time: 08:00-11:00
No books/notes/calculators etc. allowed
Examiner: Tilman Bauer

This exam consists of six problems, each worth 6 points.

Part A comprises the first two problems. The bonus points from the seminars will be automatically added to the total score of this part, which however cannot exceed 12 points.

The next two problems constitute part B, and the last two problems part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

To get full score on a problem, your solution must be well-presented and easy to follow. In particular, you should define your notation; clearly explain the logical structure of your argument in words or symbols; and motivate and explain your argument. Solutions severely lacking in these respects will achieve at most 3 points.

PART A

1. Let

$$A = \begin{bmatrix} 4 & -2 & 7 \\ 8 & -3 & 10 \end{bmatrix}.$$

(a) Find all solutions to the homogenenous system $Ax = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ **(3 p)**

(b) Find all solutions to the system $Ax = \begin{bmatrix} -5 & -3 \end{bmatrix}^T$. **(3 p)**

2. Let V be the intersection of the two hyperplanes in $\mathbb{R}^4$ given by

$$\begin{cases} x_1 + x_2 + x_3 + x_4 & = 0 \\ x_1 - x_2 - x_3 - x_4 & = 0. \end{cases}$$

(a) Find a basis for the subspace V. **(3 p)**

(b) Find a system of equations whose solutions are the subspace $W = V^\perp$. **(3 p)**

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p.t.o.

PART B

3. Let

$$A = \begin{bmatrix} 3 & a & 0 \\ 0 & 4 & 1 \\ 0 & 2 & 5 \end{bmatrix},$$

where a is a real parameter.

(a) Find the eigenvalues of A and eigenspaces corresponding to each eigenvalue. **(3 p)**

(b) For which a is A diagonalisable? **(2 p)**

(c) For $a = 0$, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. **(1 p)**

4. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

be a 3×3 matrix. Write A_{ij} for the determinant of the 2×2 submatrix arising from erasing row i and column j. Show that the following formula always holds:

$$a_{33} \det(A) = A_{11}A_{22} - A_{12}A_{21}.$$

(6 p)

PART C

5. A reflection in $\mathbb{R}^3$ is a linear map

$$r_{\vec{u}}(\vec{x}) = \vec{x} - 2(\vec{u} \cdot \vec{x})\vec{u},$$

where $\vec{u}$ is a normal vector of length 1 to the plane of reflection. Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

(a) Find a vector $\vec{u}$ of length 1 such that $r_{\vec{u}}(\vec{x})$ is a positive multiple of $\vec{e}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. **(4 p)**

(b) Show that the standard matrix of a reflection is always on orthogonal matrix. **(2 p)**

6. A matrix A is called *skew-symmetric* if $A^T = -A$. Suppose that A is a skew-symmetric $n \times n$ matrix. Determine all vectors $\vec{v}$ in $\mathbb{R}^n$ such that $(A\vec{v}) \cdot \vec{v} = 0$. **(6 p)**