KTH Teknikvetenskap

SF1626 Calculus in Several Variable
Solutions to the exam 2016-01-12

DEL A

1. Consider the function $f(x, y) = \dfrac{1}{1 + (1 - x)^2 + y^2}$.
   (a) Compute the gradient of f at the origin. **(1 p)**
   (b) Which information about the shape of the graph of f in a neighbourhood of the origin is given by the direction and the length of the gradient of f at the origin, respectively? **(2 p)**
   (c) Does the function f have a global minimum over the xy-plane? **(1 p)**

Solutions.

(a) Vi compute the partial derivatives and get

$$\frac{\partial f}{\partial x} = -\frac{2(x-1)}{(1 + (1-x)^2 + y^2)^2} \quad \text{och} \quad \frac{\partial f}{\partial y} = -\frac{2y}{(1 + (1-x)^2 + y^2)^2)^2}$$

which gives the gradient as

$$\mathbf{grad}\, f(x, y) = \left(-\frac{2(x-1)}{(1 + (1-x)^2 + y^2)^2}, -\frac{2y}{(1 + (1-x)^2 + y^2)^2} \right)$$

When substituting $(x, y) = (0, 0)$ we get

$$\mathbf{grad}\, f(0, 0) = \left(-\frac{2(0-1)}{(1 + (1-0)^2 + 0^2)^2}, -\frac{2 \cdot 0}{(1 + (1-0)^2 + 0^2)^2} \right) = \left(\frac{2}{4}, 0 \right) = \left(\frac{1}{2}, 0 \right).$$

(b) Since the function is differentiable the direction of the gradient tells us in which direction the graph of the function has the steepest upward slope. In this case it is in the positive x-direction. The length of the gradient gives the maximal directional derivative, which in this case is equal to $1/2$. This means that the the graph of f has a tangent plane that has a slope $1/2$ and this in the positive x-direction.

(c) The function is positive in the whole plane, but it can come arbitrarily close to 0. Thus there is no minimum over the whole plane.

Answer.

(a) $\mathbf{grad}\, f(0,0) = (1/2, 0)$.

(b) The function doesn't have any global minimum over the xy-plane.

2. (a) Formulate Green's Theorem in the plane including all conditions. **(2 p)**
   (b) Use Green's Theorem in order to compute the line integral

$$\int_T (x - 2x^2y)\,dx + (2xy^2 - y)\,dy$$

where T is the boundary to the parallel trapezoid with vertices in the points $(0,0)$, $(2,0)$, $(2,4)$ and $(0,5)$ traversed counter-clockwise. **(2 p)**

Solutions.

(a) If $\mathbf{F} = (P,Q)$ is a continuously differentiable vector field in the plane and D is a bounded domain with a piecewise continuously differentiable boundary curve C we have that

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dxdy$$

where C is positively oriented with respect to D.

(b) According to Green's Theorem in the plane, the line integral can be computed by means of the double integral

$$\iint_D \frac{\partial}{\partial x}(2xy^2 - y) - \frac{\partial}{\partial y}(x - 2x^2y)\,dxdy = \iint_D 2y^2 + 2x^2\,dxdy$$

where D is the given parallel trapezoid. We can describe D by the inequalities $0 \le x \le 2$ och $0 \le y \le 5 - x/2$, which makes it possible to compute the double integral by iterated integration

$$\begin{aligned}\iint_D 2y^2 + 2x^2\,dxdy &= 2\int_0^2 \int_0^{5-x/2} x^2 + y^2\,dy\,dx = 2\int_0^2 \left[x^2y + \frac{y^3}{3}\right]_0^{5-x/2} dx \\ &= 2\int_0^2 (5 - x/2)x^2 + \frac{(5-x/2)^3}{3}\,dx = 2\left[\frac{5x^3}{3} - \frac{x^4}{8} - \frac{(5-x/2)^4}{6}\right]_0^2 \\ &= 2\left(\frac{40}{3} - 2 - \frac{4^4}{6} + \frac{5^4}{6}\right) = \frac{80 - 12 - 256 + 625}{3} = \frac{437}{3}\end{aligned}$$

Answer.

(b) The value of the line integral is $437/3$.

3. The plane curve C given by the equation $27y^2 = x(x-9)^2$ can be parametrized by $\mathbf{r}(t) = (3t^2, 3t - t^3)$ where t goes through the real line.
   (a) Check that the parameter curve is part of the curve C, i.e., that the points satisfy the equation for C. **(1 p)**
   (b) Compute the velocity $\mathbf{r}'(t)$ for the parameter curve. **(1 p)**
   (c) Write down the integral in the parameter t that computes the arc length of the loop given by the interval $-\sqrt{3} \le t \le \sqrt{3}$. Simplify as much as possible. **(2 p)**

Solutions.

(a) We substitute $\mathbf{r}(t)$ in the equation and get the lefthand side

$$27(3t - t^3)^2 = 27(9t^2 - 6t^4 + t^6)$$

and the righthand side

$$3t^2(3t^2 - 9)^2 = 3t^2(9t^4 - 54t^2 + 81) = 27(t^6 - 6t^4 + 9t^2).$$

Since the left and right hand sides agree, the parameter curve satisfies the equation of C showing the the parameter curve is a subset of C.

(b) The velocity in the paramtrization is given by the derivative with respect to t. We get

$$\mathbf{r}'(t) = (6t, 3 - 3t^2)$$

(c) In order to compute the length of the curve, we integrate the length of $\mathbf{r}'(t)$. We get this length as

$$\begin{aligned} |\mathbf{r}'(t)| &= 3\sqrt{(2t)^2 + (1-t^2)^2} = 3\sqrt{4t^2 + 1 - 2t^2 + t^4} = 3\sqrt{1 + 2t^2 + t^2} \\ &= 3\sqrt{(1+t^2)^2} = 3(1+t^2). \end{aligned}$$

The length of the loop is therefore

$$\begin{aligned} \int_{-\sqrt{3}}^{\sqrt{3}} 3(1+t^2)\,dt &= \int_{-\sqrt{3}}^{\sqrt{3}} 3(1+t^2)\,dt = \left[3t + t^3\right]_{-\sqrt{3}}^{\sqrt{3}} \\ &= 3\sqrt{3} + 3\sqrt{3} - (-3\sqrt{3}) - (-3\sqrt{3}) = 12\sqrt{3}. \end{aligned}$$

Answer.

(b) $\mathbf{r}'(t) = (6t, 3 - 3t^2)$.

(c) The length of the loop is $12\sqrt{3}$ length units.

DEL B

4. Compute the double integral
$$\iint_D xy\,dx\,dy$$
where D is the domain which in polar coordinates is given by the inequalities
$$\begin{cases} 0 \le r \le \sin 2\theta, \\ 0 \le \theta \le \pi/2. \end{cases}$$
(4 p)

Solutions. When changing to polar coordinates, we get $dxdy = r\,drd\theta$. The function to be integrated becomes $xy = r^2\cos\theta\sin\theta$ and the boundary is given by the given inequalities. We can now compute the integral by iterated integration starting in the r-direction.

$$\iint_D xy\,dx\,dy = \int_0^{\pi/2}\int_0^{\sin 2\theta} r^3\sin\theta\cos\theta\,drd\theta = \int_0^{\pi/2}\left[\frac{r^4}{4}\sin\theta\cos\theta\right]_0^{\sin 2\theta} d\theta$$
$$= \int_0^{\pi/2}\frac{(\sin 2\theta)^4}{4}\frac{\sin 2\theta}{2}\,d\theta = \frac{1}{8}\int_0^{\pi/2}(\sin 2\theta)^5\,d\theta = \frac{1}{8}\int_0^{\pi/2}(1-\cos^2 2\theta)^2\sin 2\theta\,d\theta$$

We can make the change of variables $t = \cos 2\theta$ and we get $dt = -2\sin 2\theta\,d\theta$, which gives

$$\frac{1}{8}\int_0^{\pi/2}(1-\cos^2 2\theta)^2\sin 2\theta\,d\theta = -\frac{1}{16}\int_1^{-1}(1-t^2)^2\,dt = \frac{1}{16}\int_{-1}^{1} 1 - 2t^2 + t^4\,dt$$
$$= \frac{1}{16}\left[t - \frac{2t^3}{3} + \frac{t^5}{5}\right]_{-1}^{1} = \frac{2}{16}\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{2}{16}\left(\frac{15-10+3}{15}\right) = \frac{1}{15}.$$

Answer. $\displaystyle\iint_D xy\,dxdy = \frac{1}{15}$

5. (a) Let $f(x, y)$ be a function of two variables. Explain what it means that a point (x_0, y_0) is a critical point, local maximum and local minimum, respectively. **(2 p)**

   (b) The function $f(x, y) = e^{x-x^3/3-y^2}$ has critical points in $(1, 0)$ and $(-1, 0)$. Determine whether these are local maxima, local minima or neither. **(2 p)**

Solutions.

(a)
- A *critical point* is a point where the gradient is zero.
- A *local maximum* is a point where the function attains a value that is maximal in some disk with center at the point.
- A *local minimum* is a point where the function attains a value that is minimal in some disk with center at the point.

(b) In order to check whether they are local maxima or minima, we look at the second order derivatives. The quadratic form in the Taylor polynomial of degree two is given by the matrix

$$\begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -2x & 0 \\ 0 & -2 \end{bmatrix} e^{x-x^3/3-y^2}$$

In the point $(1, 0)$ we get the matrix

$$\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} e^{2/3}$$

which is negative definite, which shows that the function has a local maximum at the point $(1, 0)$.
For the point $(-1, 0)$ we get the matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} e^{-2/3}$$

which is indefinite and the point $(-1, 0)$ is a saddle point and hence neither a maximum nor a minimum.
The quadratic forms in the two points are given by $-(h^2 + k^2)e^{2/3}$ and $(h^2 - k^2)e^{-2/3}$, respectively.

Answer.

(b) $(1, 0)$ is a local maximum and $(-1, 0)$ is neither a local maximum nor a local minimum.

6. The curve C is a connected part of the hyperbola $xy = 1$ from $(1,1)$ to the point P. Determine P when
$$\int_C (2x+y)\,dx + (x-8y)\,dy = 3.$$
(4 p)

Solutions. Let $(a, 1/a)$ be the unknown point P. We can parametrize the curve as $\mathbf{r}(t) = (t, 1/t)$ and we get $d\mathbf{r} = (t, -1/t^2)\,dt$. Hence the line integral is given by

$$\int_C (2x+y)\,dx + (x-8y)\,dy = \int_1^a (2t + \frac{1}{t})\,dt + (t - \frac{8}{t})(-1/t^2)\,dt$$
$$= \int_1^a 2t + 8t^{-3}\,dt = \left[t^2 - 4t^{-2}\right]_1^a = a^2 - 4a^{-2} - 1 + 4 = 3 + a^2 - 4a^{-2}.$$

The condition gives the equation
$$a^2 - 4a^{-2} = 0$$
which can be written as $a^4 - 4 = 0$ since a cannot be zero. This equation has the four solutions $a = \pm\sqrt{2}$ and $a = \pm i\sqrt{2}$. We are looking for the positive real solution and hence we get $a = \sqrt{2}$. This means that the point P is given by $P = (\sqrt{2}, 1/\sqrt{2})$.

An alternative way of solving the problem is to see that the vector field $\mathbf{F}(x,y) = (2x+y, x-8y)$ is conservative and we can find a potential by first integrating in the x-direction to get $\Phi(x,y) = x^2 + xy + g(y)$. Differentiation with respect to y now gives $x + g'(y) = x - 8y$, and we can choose $g(y) = -4y^2$. The value of the integral can now be computed as a difference in the potential, i.e,
$$\int_C (2x+y)\,dx + (x-8y)\,dy = \Phi(P) - \Phi(1,1).$$
Since $\Phi(1,1) = 1^2 + 1\cdot 1 - 4\cdot 1 = -2$ we look for $P = (a, 1/a)$ with $\Phi(a, 1/a) = 1$. This condition gives the same equation for a as before.

Answer. The point is $P = (\sqrt{2}, 1/\sqrt{2})$.

DEL C

7. Consider the equation $F(x,y)=0$ where $F(x,y)=xe^y+ye^x$.
   (a) Show that there is a function g with $g(0)=0$ such that $F(x,g(x))=0$ for x close to 0. **(1 p)**
   (b) Compute the Taylor polynomial of degree two for g at $x=0$. **(3 p)**

Solutions.

(a) We can use the implicit function theorem in order to show this. The requirements are satisfied since the functions involved are continuously differentiable. We need to check that the partial derivative of F with respect to y is non-zero in order for us to be able to solve y as a function of x.

$$\frac{\partial F}{\partial y}=xe^y+e^x$$

And hence

$$\frac{\partial F}{\partial y}(0,0)=0\cdot e^0+e^0=0+1=1\neq 0.$$

The implicit function theorem now tells us that we can solve y as a function $y=g(x)$ in a neighborhood of the origin.

(b) In order to compute the Taylor polynomial of degree two, we can differentiate the identity identiteten $F(x,g(x))=0$ with respect to x. We get

$$0=\frac{d}{dx}\left(xe^{g(x)}+g(x)e^x\right)=e^{g(x)}+xe^{g(x)}g'(x)+g'(x)e^x+g(x)e^x$$

and when substituting $x=0$ we get

$$0=1+g'(0)+0$$

which gives $g'(0)=-1$. In order to compute the second derivative, we differentiate once more and get

$$\begin{aligned}0=\frac{d}{dx}&\left(e^{g(x)}+xe^{g(x)}g'(x)+g'(x)e^x+g(x)e^x\right)\\ &=e^{g(x)}g'(x)+e^{g(x)}g'(x)+xe^{g(x)}(g'(x))^2+xe^{g(x)}g''(x)\\ &=+g''(x)e^x+g'(x)e^x+g'(x)e^x+g(x)e^x.\end{aligned}$$

When substituting $x=0$ we get

$$0=g'(0)+g'(0)+0+0+g''(0)+g'(0)+g'(0)+0$$

which gives $g''(0)=-4g'(0)=4$. The Taylor polynomial of degree two is given by

$$p(x)=0-x+4x^2/2=-x+2x^2.$$

Answer.

(b) The Taylor polynomial of degree two is $p(x)=-x+2x^2$.

8. The function f is given by

$$f(t) = \iint_D \exp\left(\frac{tx}{y^2}\right)\, dxdy$$

where $t > 0$ and the domain D is defined by $t \leq x \leq 2t$ and $t \leq y \leq 2t$. Show that

$$f(t) = Ct^2$$

for some constant C. **(4 p)**

Solutions. We make the change of variables $u = x/t$, $v = y/t$ and get the Jacobi matrix

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} 1/t & 0 \\ 0 & 1/t \end{bmatrix}$$

and hence $dudv = 1/t^2 dxdy$. Therefore we have

$$f(t) = \int_1^2 \int_1^2 \exp\left(\frac{t^2 u}{t^2 v^2}\right) t^2 dudv = t^2 \int_1^2 \int_1^2 \exp\left(\frac{u}{v^2}\right)\, dudv$$

Thus we have shown that $f(t) = Ct^2$ where the constant C is given by

$$C = \int_1^2 \int_1^2 \exp\left(\frac{u}{v^2}\right)\, dudv.$$

9. Let $g(r)$ be a twice continuously differentiable function with $g'(4) = 1$. Compute the integral

$$\iint_D \left(f''_{xx} + f''_{yy}\right)\, dxdy,$$

where $f(x,y) = g\left(x^2+y^2\right)$ and D is a circular disk with radius 2 centered at the origin.

(4 p)

Solutions. Since the requirement for the divergence theorem in the plane are satisfied, we can rewrite the double integral as a flux integral

$$\iint_D \left(f''_{xx} + f''_{yy}\right)\, dxdy = \int_C \mathbf{grad}\, f \cdot d\mathbf{N}$$

We ge the gradient of f as

$$\mathbf{grad}\, f = \left(2xg'\left(x^2+y^2\right), 2yg'\left(x^2+y^2\right)\right)$$

A normalized normal vector to the circle C is given by $\mathbf{N}(x,y) = \frac{1}{2}(x,y)$ and hence the line integral can be computed as

$$\int_C \mathbf{grad}\, f \cdot d\mathbf{N} = \int_C g'(4)(2x,2y)\cdot(1/2x, 1/2y)\, ds = \int_C (x^2+y^2)\, ds = 4\int_C ds = 4\cdot 4\pi = 16\pi.$$

Another way to solve the problem is to compute $f''_{xx} + f''_{yy}$ as

$$f''_{xx} + f''_{yy} = 4g'\left(x^2+y^2\right) + 4(x^2+y^2)g''\left(x^2+y^2\right).$$

A change to polar coordinates then gives

$$\iint_D \left(f''_{xx} + f''_{yy}\right)\, dxdy = \int_0^{2\pi}\int_0^2 4g'(r^2) + 4r^2g''(r^2)\, rdrd\theta = 2\pi\int_0^2 4g'(r^2) + 4r^2g''(r^2)\, rdr$$

The substitution $t = r^2$ then yields $dt = 2r\, dr$ and

$$2\pi\int_0^2 4g'(r^2) + 4r^2g''(r^2)\, rdr = 4\pi\int_0^4 g'(t) + tg''(t)\, dt = 4\pi\left[tg'(t)\right]_0^4 = 4\pi\cdot 4 - 4\pi\cdot 0 = 16\pi.$$

Answer. $\iint_D \left(f''_{xx} + f''_{yy}\right)\, dxdy = 16\pi.$